Attitudes to Limiting Cases in Ancient Logic

David Makinson

Abstract
We draw attention to a tendency of ancient Peripatetic logic echoing into the middle ages and beyond, to disregard limiting cases. Specific instances of this attitude have often been noticed, but they do not seem to have been taken together as illustrating a general pattern. We review the variety of ways in which the tendency manifested itself, discuss its apparent absence or at least reduced presence in Stoic logic, note some exceptions, and speculate on the reasons for its existence. The note raises as many questions as it answers, and should be seen as an invitation to further discussion.

Keywords: Limiting cases, Aristotelian logic, Peripatetic logic, Stoic logic, partitions, generation.

1. Limiting cases in Peripatetic logic

This note discusses contrasting attitudes to limiting cases in ancient logic.[1,2] We suggest that there was a tendency in Peripatetic logic to disregard limiting cases while it would appear, from the little evidence available, that for Stoic logicians the tendency was to include them. We begin by reviewing examples of each, as well as exceptions; in a later section we speculate on possible reasons.

The Peripatetic attitude to limiting cases shows itself in several different ways, arising in both semantic and syntactic contexts.

(a) On the semantic level, a much discussed manifestation concerns the values that may be taken by a term in a categorical proposition. It was allowed that there may be many items falling under the term or only a few but – on the usual interpretation of Aristotle since the publication of Łukasiewicz 1957 – at least one. In other words, on this reading, terms are never void: the limiting case of emptiness is left aside. Some medieval logicians experimented with it, but mainly it was ignored until the second half of the nineteenth century.

To be sure, that interpretation of Aristotle has not gone unchallenged. In particular, Read 2015 argued that he was quite willing to allow empty terms in categorical propositions, and proposed a representation in the language of first-order logic that admits them while preserving intact the traditional square of opposition. However, even under Read's interpretation, the fact remains that neither Aristotle nor his ancient followers gave empty terms much attention, largely ignoring this limiting case.

(b) Another semantic manifestation of the disregard for limiting cases is the Peripatetic tendency, when discussing genera and species, to think of the whole-part relation between kinds as what we would now call proper inclusion. That is, one kind was taken to be a 'part' of another just if everything of the former kind is of the latter one but not conversely. The limiting case that the converse also holds, so that exactly the same items are of the two kinds, was typically excluded from the relation. In other words, proper inclusion was seen as basic as a relation between kinds; inclusion as it is known today was treated as secondary (see e.g. a passage from the *Topics* cited in Kneale & Kneale 1962 pp. 36-7).

On the other hand, when discussing a related question in syllogistic, Aristotle and his followers stood resolutely in favour of an opposite policy. They took propositions of the form *Some S are P* as meaning *Some-and-perhaps-all S are P*, thus including the limiting case that *S* holds of the same items as *P*, so that to express *Some-but-not-all S are P*, they needed to combine the forces of particular affirmative and particular negative. This contrast of policies makes good sense, showing flexibility rather than incoherence. For while excluding coextensive kinds from the whole-part relation simplifies the treatment of classification, including coextensive terms within the compass of s*ome* leads to a more elegant theory of inference.

(c) An important syntactic manifestation of the neglect of limiting cases is that the Peripatetics rarely envisaged the identification of terms as instances of their paradigm forms of deduction. Recall that Aristotle made use of schematic letters for terms, to create the familiar forms *All S are P*, *No S are P*, *Some S are P*, *Some S are not P*. The letters could be instantiated in two ways: either to specific terms, giving complete propositions, or to other letters. For example, the letters in *All S are P* could be instantiated by the terms *man* and *animal* to yield the specific proposition *All men are animals*, or instantiated by other letters to yield, say, *All Q are R*. But almost all examples of such instantiation in the surviving literature of Peripatetic term logic are *injective*; in other words, distinct term letters in a propositional or inferential form were not identified. For example, one did not count the proposition *All men are men* among the instances of *All S are P*, nor was the form *All S are S* considered among the schematic instances. In syllogistic inference, the inference form *All S are P, all P are Q, so all S are Q* was not in general thought of as covering such a form as *All S are P, all P are S, so all S are S.*

However, as pointed out to the author by Paul Thom, there is an exception to this pattern in the writings of Aristotle himself. Book 2 chapter 15 of the *Prior Analytics* discusses inferences with 'opposed premises' and recognizes that in some cases they can be valid. Aristotle gives two examples, of which one is *Some medicine is science, no medicine is science, so some science is not science*, while the other is the same but with a universal rather than particular affirmative premise. Both are seen as instantiating canonical syllogistic forms with three terms. Nevertheless, this isolated passage seems to have had little traction in the subsequent Peripatetic tradition.

(d) In general, the Aristotelians did not take into consideration inferences in which the conclusion is the same as one of the premises. The late Roman author Boethius did, at one point, consider such an inference, but called it a *syllogismus perridiculus* while the medieval philosopher Abelard, in a commentary, called it a *syllogismus ridiculus* (Thom 2010, p. 232).

(e) The *Institutio Logica*, usually attributed to Galen in the second century CE, declines to use the term 'conjunction' in the limiting cases that the propositions jointly asserted are incompatible with each other, or one of them is a logical consequence of the other (see Mates 1961 p. 118). However, that leaves open the question whether Galen would have accepted the possibility of argument in which the premises contradict each other, when those premises are not combined in a conjunction.

There is reason to think that many Peripatetics would have been open to that possibility. For, as mentioned in point (c) above, Aristotle did accept an example of an argument with 'opposed premises', albeit in an isolated passage. More significantly, Greek mathematics made use of *reductio ad absurdum* as a method of proof, and Aristotle employed it in his reductions of certain syllogisms to the first figure. So it seems reasonable to take Peripatetics as typically allowing one to construct arguments with incompatible (but not conjoined) premises when one's purpose is to discover a hidden inconsistency among the premises and use that to reject one among them. To this extent, the Peripatetic attitude to this limiting case is nuanced.

2. Limiting cases in the Stoic tradition

It is not clear whether Chrysippus and other early Stoics knew, or cared to know, of Aristotle's term-logic (Barnes 2012a,b). In any case, the scattered fragments about Stoic logic that remain deal only with their distinctive focus on what we would now call propositional logic. We will therefore consider their attitude to limiting cases in that domain.

(a) It is clear from the accounts of Sextus Empiricus and Diogenes Laertius that the Stoics were quite happy with the identification of sentence-signs in their logical forms; in particular, they worked with what were called 'duplicated conditionals' – propositions of the form *If p then p*. For example, according to Sextus Empiricus in book VIII of *Against the Mathematicians*:

> ... molecular propositions … are composed from two occurrences of the same proposition or from different propositions, and are composed by means of a connective or connectives. For example, 'If it is day, it is day' … 'It is day and it is light', 'It is day or it is night' (Mates 1961 p. 96).

Diogenes Laertius tells us the same, almost verbatim in book VII of *Lives of Eminent Philosophers* (see Mates 1961 pp. 112-3). The explicit reference to 'the same proposition' suggests that this was considered a notable feature. Examples are given, with such duplicated conditionals serving as premises for applications of both modus ponens and proof by cases (disjunctive proof) (see Mates 1961 pp. 66, 81). Sextus Empiricus also gives an example of a derivation from the five Stoic 'undemonstrable' argument forms, that makes use of the form *If p then if p then r* (Mates 1961 pp. 78-9). This is not quite a 'duplicated conditional' but it can be thought of as limiting the generality of *If p then if q then r* by identifying two of the three variables.

(b) The passage of Galen's *Institutio Logica* mentioned above, refusing to countenance conjunctions made from propositions that are mutually incompatible or where one is a logical consequence of the other, also complains that 'the followers of Chrysippus' had no compunctions in this regard (see Mates 1961 p. 118).

(c) The Stoic attitude to arguments in which the conclusion coincides with one of the premises seems to have been more liberal than that of the Peripatetics. Alexander of Aphrodisias criticized the Stoics for accepting such arguments, but the clearest example that he gives is, at the same time, an instance of modus ponens with a duplicated conditional as major premise: *If p then p, but p, so p* (see Mates 1961 pp. 66, 125-7). So we know that the Stoics accepted *some* inferences in which the conclusion coincides with a premise, namely those that instantiate other inference patterns lacking that property, but some uncertainty remains whether they accepted *all* such inferences.

(d) Finally, there is an exception to the trend we are describing, for there is a context in which the Stoics gave pride of place to a notion that *excludes* a limiting case. Recall that an exclusive disjunction comes out true just when exactly one of its two disjuncts is true; it is false in the limiting case that both the disjuncts are true. While the Stoics recognized the existence of inclusive disjunction (and also, it seems, of some non-truth-functional kinds), the exclusive version was given priority. In particular, it is the only form of disjunction that appears in the five Stoic 'undemonstrated' argument forms, figuring in two of them. One of those two argument forms would, indeed, fail for inclusive disjunction.

3. A missing witness

We have not mentioned what is sometimes considered the most salient example of a limiting case in logical theory – the validity of any argument from inconsistent premises to an arbitrary conclusion, traditionally known as *ex contradictione sequitur quodlibet* or, more briefly (and misleadingly) as *ex falso quodlibet*, in recent times also dubbed *explosion* or, more specifically, *right explosion*.

The reason is simple: scholars have no direct knowledge of ancient attitudes to this principle, whether formulated for the validity of an argument, as above, or for the logical truth of a corresponding conditional proposition. The earliest surviving discussions of it date from the middle ages.

Regarding the ancient Peripatetics, there is no evidence to suggest that any of them accepted explosion as a valid form of argument, and they may not have even articulated it for consideration. As mentioned in point (e) of section 1, Aristotle and his followers were ready to admit *reductio ad absurdum* as a mode of argument, and thus accept at least some inferences from inconsistent premises. But that is a far cry from explosion.

As for the Stoics, It would be unsafe to infer that they rejected explosion from the fact that in the surviving texts their critics never take them to task for advocating it (*pace* Priest 1998 p. 41, 2007 p. 132). Other hypotheses are consistent with this situation, for example that the principle was discussed in texts that are not among the few fragments that have come down to us, or that neither the Stoics nor their critics even formulated it.

On the other hand, it would be hazardous to conclude that the Stoics accepted explosion on the ground that they had the formal means to do so in just a few steps from material that they recognized. What is now referred to as the Lewis derivation of explosion dates back to Alexander Neckham and William of Soissons in the twelfth century (see e.g. Martin 1986, 2009), but all the individual steps in that derivation, as well as a ‘cut’ rule for chaining those steps into a whole, appear to have been known already to the Stoics (see e.g. Tkaczyk 2024a,b). Moreover, as Tkaczyk shows in detail, there are a number of other ways in which the Stoics could legitimately have derived explosion using principles and ‘themata’ (rules of derivation) that the ancients attributed to them.[3] However, the admissibility, even the triviality, of those steps does not mean that they were actually carried out.

On the side of rejection, appeal might be made, to a passage in Sextus Empiricus’ *Outlines of Pyrrhonism* that distinguishes between four different Stoic conceptions of conditional propositions (see e.g. Mates 1961 pp. 47-8 or Kneale & Kneale 1962 pp. 128-9). Sextus tells us that the third and fourth of these conceptions treat the conditional statement *If atomic elements of things do not exist then atomic elements of things do exist* as false, despite the fact that the antecedent is false (according to the Stoics) and the consequent true independently of any reference to time (an aspect important to the Stoics). Sextus adds that, under the third conception, for a conditional to be counted as true there must be some ‘connection’ between antecedent and consequent while, for the fourth conception the antecedent should ‘include’ the consequent – in some unexplained senses of those two terms. This passage might be said to show that at least some Stoics, namely those following the third and fourth conceptions of the conditional, rejected explosion.

However, such a reading seems rather hasty. Ancient writers may well have accepted or rejected conditional statements like the one mentioned by Sextus without ever conceptualizing the general principle of explosion as we understand it today. Sextus’ passage gives no names, and it is not clear whether those of the third and fourth conceptions were major or minor figures compared to those of the first and second kinds. Finally, Sextus wrote in the second century AD, by which time a certain syncretism between Stoic and Peripatetic perspectives was emerging, so that the third and fourth conceptions could have had a mixed heritage.

In conclusion, it seems that while one may see the ancient Peripatetics as dismissing, or more likely not even considering the principle of explosion, it would be unsafe on the meagre evidence available to take a definite view regarding the Stoic attitude (or attitudes) to it.

4. Why did the Peripatetics avoid limiting cases?

The Aristotelians did not so much *argue against* recognizing limiting cases as *ignore* them, thus leaving them invisible to the reader. Hence, one can only speculate on their reasons. We review a number of possible factors of varying generality, which need not exclude each other.

(a) The present author suspects that an important factor lies in an aspect of Aristotelian methodology that has been graphically described by Barnes:

> There was a distinctively Peripatetic way of doing formal logic … The main feature of that approach is its use of the method of exhaustive survey: a general form of argument is isolated; its various subforms are classified; and each possible instance of those subforms is examined *seriatim*, it being determined whether the instance is valid or invalid (Barnes 2012b p. 427).

Put rather more formally and generally, the method consisted of four steps. (1) Pose the problem by identifying the domain of items under investigation and fixing a highly valued property that holds for only some of those items; the question then is to determine which of them possess it. (2) Partition the domain into a finite number of cells. (3) Identify a distinguished sub-collection of those cells in such a way that one can check, by exhaustive enumeration, that the valued property holds of exactly the items that fall into one of the distinguished cells. (4) Tidy up.

In the case of logic, the items are arguments, the desirable property is validity, and the partition is into a finite number of forms. Aristotle used the procedure principally for what we now call syllogistic, but also for his incipient modal logic (see e.g. Smith 2022 section 5.6). Later Peripatetics also applied it when ‘stagirizing’ Stoic ideas in an attempt to build their own logic of ‘hypothetical syllogisms’ (cf. Bobzien 2002 section 5). Tidying up consisted of finding connections, simplifications and complements. Connections were already noted by Aristotle when he reduced second and third syllogistic figures to the first one (Smith 2022 section 5.5). Simplifications providing summary criteria for validity appeared gradually: Aristotle already declared certain conditions to be necessary for validity (see again Smith 2022 section 5.5) but short lists purporting to be both necessary and jointly sufficient for validity are first reported from the late fifteenth century for specific figures and the early seventeenth century for all figures together (Kneale & Kneale 1962 pp. 272-3). Complements included diagrams such as the square of opposition, which go back to Apuleius or before, became popular in the middle ages, and remain so today (cf. Makinson 2024).

It is an impressive methodology. However, it is saddled with two important constraints. One, which does not concern us here, is that the partition is *finite* and, indeed, must be reasonably small for the third step to be practicable by hand. Another constraint, which is relevant to the treatment of limiting cases, is that each cell of the partition is *uniform with respect to the property in question*. That is, either *all* items in the cell have the property, or *none* do.

Limiting cases tend to disturb that assumption. Sometimes, the property may hold for principal cases within a cell, but not for a limiting one. For example, at least from the standpoint of the usual first-order translations of categorical propositions, this happens for the property of validity, the cell consisting of affirmative subalternations, that is, inferences of the form *All S are P, so some S are P*, and the limiting case that *S* is void. On other occasions, the property may fail for principal cases within a cell, yet hold for a limiting one. For example, this happens for the property of validity, the cell consisting of conversions of the universal affirmative, that is, inferences of the form *All S are P, so all P are S*, and the limiting case that subject and predicate are the same term.

In brief, limiting cases can be difficult to integrate into the Aristotelian methodology of ‘partition and rule’ because they upset the uniformity of some of the cells. To restore uniformity, it is tempting to ignore the troublesome cases.

In Stoic logic, the typical method of investigation seems to have been quite different, based on *generation* rather than on *partition*. The Stoics generated their propositional schemas from atomic elements by the iterable application of connectives, and generated valid argument forms from five ‘undemonstrable’ schemes by the iterable application of four rules of derivation (see e.g. Bobzien 2020). In effect, they gave rudimentary recursive constructions, just as do modern logicians. Limiting cases do not cause problems for this procedure.

(b) Recall that for Aristotle and his followers, validity was not the only criterion for assessing arguments. In the *Topics* and *Prior Analytics*, Aristotle uses a term generally translated as ‘demonstration’ for an argument which as well as being valid, has premises that are both ‘true and primary’ (Bocheński

1961 pp. 44-45). The latter notion is obscure, to say the least; but Alexander of Aphrodisias in the late second or early third century CE expressed the desired features in a more cognitive/epistemic manner: the argument should 'make something clear that does not appear to be known, and to do this by means of what is known and clear' (Mates 1961 p. 66).

Alexander's criteria immediately rule out some of the limiting cases we have described, namely those where: (i) the conclusion coincides with a premise (for one should be unknown and the other known), (ii) the premises are mutually inconsistent (since inconsistent premises cannot be known, even if they may sometimes be believed), and perhaps the case where (iii) the conclusion is an absolutely trivial logical truth (for then it would already be known to everybody). If one is interested primarily in arguments that satisfy such conditions, then valid ones that cannot do so may appear unworthy of serious attention.

(c) On a mundane level, in ordinary discourse, we reason with limiting cases less frequently than with principal ones. For example, outside of *reductio ad absurdum*, we rarely use premises that we already know to be inconsistent (but see Makinson 1965 for a possible exception). We seldom reason with terms that we know to be empty, or argue to a conclusion that we realize is trivially true. The Peripatetics thus had a pragmatic reason for neglecting limiting cases, comparable with that of contemporary logicians when, in first-order logic, they disregard the empty domain of discourse.

(d) The neglect of empty terms perhaps also reflected an aversion, widespread among Greek thinkers, to anything associated with nothingness. Plato's dialogue *The Sophist* raises the question of whether it is possible to speak coherently about non-being. Many philosophers rejected the possibility of empty space; in particular, in his *Physics* Aristotle had explicit arguments to this effect. Greek mathematicians were reluctant to admit zero to the club of numbers, or even to give it a special symbol as name, and in the *Physics* again, Aristotle argued against its acceptance on the ground that one cannot divide a whole number by zero. In geometry, a line was never of zero length, nor an angle of zero degrees. Disregard of empty terms was thus in good company.

(e) Finally, a far-fetched speculation. A recurring theme in Greek philosophy of life, including that of Aristotle, was that the virtuous man always follows a middle way and avoids extremes. Now, limiting cases in logic can be thought of as extremes, so it might have been felt, as a state of mind rather than as a conscious conclusion, that the responsible logician should leave them out of consideration.

Acknowledgements
Thanks to Antonielly Garcia Rodrigues, Wolfgang Lenzen, Paul Thom, Marcin Tkaczyk and an anonymous reviewer for very helpful comments and corrections.

Notes

1. By a ‘limiting case’, we mean a special case of an abstract object that arises when it, or one of its components, takes on an extreme value under some natural ordering. Contemporary mathematicians also speak of ‘exceptional’ or ‘edge’ cases and, when the case is treated with distaste, may describe it as ‘degenerate’or ‘pathological’ (cf. Wikipedia 2024).

2. By ‘ancient logic’ we mean logic in Greek and Roman antiquity, dominated by the Stoic and Peripatetic traditions; for convenience we include Aristotle himself among the latter. For a brief review of these traditions, some of the personalities involved and pointers to literature, see e.g. Bobzien 2020. For more extended overviews see e.g. Bocheński 1961 (part II), Kneale & Kneale 1962 (chapters II-IV), and the essays on special topics in Castagnoli & Fait 2023.

3. As Tkaczyk observes, the most startling way in which the Stoics could legitimately have derived explosion takes just two steps: given the validity of the argument form *p, not:q, so p*, we have the validity of the form *p, not:p, so q*, The ‘given’ is an instance of the general reflexivity principle *p, q, so p*, which appears to have been accepted by the Stoics (see point (c) of section 2 above). The passage from that to explosion applies the rule of antilogism (also known as transposition): it tells us that whenever *p, q, so r* is valid, so too is *p, r*, so q**, where the starred letters represent any propositions in contradictory relation to the unstarred ones. This rule was attributed to the Stoics by Apuleius in a logical work under the misleading title *On Plato*, where it is formulated very clearly as follows: ‘If two sentences entail a third sentence, then either of the two sentences together with the contradictory of the third sentence entails the contradictory of the other of the two sentences’ (Tkaczyk 2024a p. 114, 2024b p. 8).

References

Barnes, J. 2012. *Logical Matters: Essays in Ancient Philosophy II*, Oxford: Clarendon Press.

Barnes, J. 2012a. Aristotle and Stoic logic. In Barnes 2012 pp. 382-412. This is a revised version of an article with the same title that first appeared in K. Ierodikonou (ed) *Topics in Stoic Philosophy*, Oxford: Clarendon Press, 1999, pp. 23-53.

Barnes, J. 2012b. Theophrastus and Stoic logic. In Barnes 2012 pp. 413-432. This is a revised version of an article with the same title that first appeared in J. Wiesner (ed.) *Aristoteles Werk und Wirkung—Paul Moraux gewidmet I*, Berlin: Walter de Gruyter, 1985, pp. 557-576.

Bobzien, S. 2002. The development of modus ponens in antiquity: from Aristotle to the 2nd century AD. *Phronesis* 47: 359-394.

Bobzien, S. 2020. Ancient logic. The *Stanford Encyclopedia of Philosophy*. https://plato.stanford.edu/entries/logic-ancient/. Accessed 25 May 2024.

Bocheński, I.M. 1961. *A History of Formal Logic*, Notre Dame, Indiana: University of Notre Dame Press.

Castagnoli, L. & P. Fait (eds) 2023. *The Cambridge Companion to Ancient Logic,* Cambridge: Cambridge University Press.

Kneale, W. & M. Kneale 1962. *The Development of Logic,* Oxford: Clarendon Press.

Łukasiewicz 1957. *Aristotle's Syllogistic*, second enlarged edition, Oxford: Clarendon Press.

Makinson, D. 1965. The paradox of the preface. *Analysis* 25: 205-207

Makinson, D. 2024. Display conventions for octagons of opposition. *Axioms* 13, 287. https://doi.org/10.3390/axioms13050287

Martin, C.J. 1986. William's machine. *The Journal of Philosophy* 83: 564-572.

Martin, C.J. 2009. The development of logic in the twelfth century. In R. Pasnau (ed.) *The Cambridge History of Medieval Philosophy,* Cambridge University Press, pp. 129-146.

Mates, B. 1961. *Stoic Logic* (second edition), Berkeley: University of California Press.

Priest, G. 1998. What is so bad about contradictions? *The Journal of Philosophy* 95: 410-26.

Priest, G. 2007. Paraconsistency and dialetheism, in D. Gabbay & J. Woods (eds) *Handbook of the History of Logic Vol 8*, Amsterdam: Elsevier, pp. 129-204.

Read, S. 2015. Aristotle and Łukasiewicz on existential import. *Journal of the American Philosophical Association* 1: 535-544.

Smith, R. 2022. Aristotle's logic. The *Stanford Encyclopedia of Philosophy*. https://plato.stanford.edu/entries/aristotle-logic/#SylMod. Accessed 13 June 2024.

Thom, P. 2010. Three conceptions of formal logic. *Vivarium* 48: 228-242.

Tkaczyk, M. 2024a. Are ancient logics explosive? *History and Philosophy of Logic* 45: 109-123.

Tkaczyk, M. 2024b. The principle of explosion in the Stoic logic. *Logic and Logical Philosophy*, online first at DOI: 10.12775/LLP 2024 012.

Wikipedia 2024. https://en.wikipedia.org/wiki/Limiting case (mathematics). Accessed 13 June 2024.

David Makinson
School of Historical and Philosophical Inquiry
University of Queensland, Australia
d.makinson@uq.edu.au