Intuitionism Without Intuition: Against the Phenomenological Account

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Abstract

In this paper, we will consider the impact of the *intuitionistic philosophical program* on the *intuitionistic mathematical program*. In particular, we will concentrate on the phenomenological approach to the philosophy of intuitionism. We shall argue that recent attempts (such as that of Mark van Atten) to justify intuitionistic mathematics by appealing to Husserlian phenomenology are seriously contributing to the failure of the intuitionistic mathematical program. Our claim is that one of the main reasons for the failure of the intuitionistic mathematical program lies in the emphasis that is given to the philosophical program. Our thesis will be illustrated by an example: the phenomenological justification of the intuitionistic notion of choice sequence.

As a matter of fact, not much intuitionistic mathematics has been produced so far and, in this sense, we might say that the intuitionistic mathematical program is failing. Nevertheless, we think that the intuitionistic mathematical program might be defensible, but it should be defended directly, by the actual production of significant pieces of intuitionistic mathematics, rather than trying to legitimate its entities and principles of reasoning by appealing to philosophical theories.

It should be clear from the start that it is beyond the scope of this paper to criticize in detail the phenomenological approach to intuitionism. Our criticism is more of a methodological nature, and it amounts to saying that it does not appear to be a good strategy to require from mathematicians that they embrace phenomenology before actually doing intuitionistic mathematics.

1. Intuitionistic Mathematics and Philosophy

On a fairly acceptable reading, Brouwerian intuitionism is a *mathematical program*: it is about doing mathematics in a constructive way, approximately meaning without resorting to arbitrary conceptualizations, such as, for example, postulating the possibility of choosing an element among an

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uncountably infinite number of elements, without indicating how to make such a choice. On a more grandiose reading, one that in some sense has become the default reading, Brouwerian intuitionism is first of all a *philosophical program*, a program issued from a comprehensive philosophical idea that has a bearing on a wide range of topics of philosophical significance, one of which is mathematics. A good example of such a view is the following statement (van Dalen 1998: 212):

> Brouwer, from 1905 onwards, elaborated a philosophy, not just of mathematics, but an overall one, on the basis of a revised idealism in the sense of Kant. What distinguishes Brouwer from his fellow philosophers of mathematics, is that his philosophy is much more ambitious; Brouwer presents a full-blown philosophy that covers not just mathematics and science, but also epistemology, ethics, social philosophy.

Even allowing that Brouwerian intuitionism might be correctly intended as a *full-blown philosophy*, we think that nobody could deny that its outcome should be intended as supporting the development of intuitionistic mathematics.

With these two things both granted—namely, that intuitionism is first of all a philosophy and that it should be intended as promoting intuitionistic mathematics—let us now take note that no matter how much effort has been and is spent in further developing philosophical intuitionism, only a very restricted number of mathematicians adhere to the intuitionistic mathematical program by actively contributing to its development. This is a fact. The provocative thesis we shall be defending is that the view that *intuitionism is (first of all) a philosophy* not only does not actually foster the development of intuitionistic mathematics but rather does the contrary, namely it makes much easier for the supporters of classical mathematics to dismiss intuitionistic mathematics. This is particularly evident in the case of the phenomenological defense of intuitionism. In fact, as we shall argue later, the phenomenological approach to intuitionism offers a very simple way of rejecting intuitionistic mathematics.

If we were to learn intuitionistic mathematics just as mathematics, we would be straightforwardly introduced to its objects, principles of reasoning, axioms, rules, and so on, just as it happens for classical mathematics and set theory. Perhaps we would not even come to think that the entities of intuitionistic mathematics might not be really *objects*; we would probably just learn how to make mathematical reasonings with them and eventually would evaluate the conclusions they allow us to make, focusing on whether intuitionistic mathematics is bringing us new, useful, mathematical insight. Nevertheless, the thesis that intuitionism is first of all a philosophy demands that before we do any mathematics at all, we first learn the underlying philosophical arguments. These arguments have the aim of introducing and justifying the entities of intuitionistic mathematics and the way we reason about them. The claim is that in this case, unlike in the case of classical mathematics, a philosophical justification is needed, precisely because the entities of intuitionistic mathematics differ from the usual mathematical entities. In Mark van Atten's words (van Atten 2003: 4):

> Most mathematicians have been skeptical about Brouwer's solution from the beginning. The objects in classical mathematics—numbers, geometrical objects, sets—seem to be outside of space and time, never to change, and to exist independently of acts of any subject. Choice sequences behave in just the opposite way: they grow in time, according to the choices a subject makes. With such credentials, surely choice sequences should never be allowed into the mathematical universe—or should they? [...] I think that they should. *However, that claim needs a philosophical justification* (my emphasis).

A philosophical argument, therefore, is needed in order to legitimize the entities of intuitionistic mathematics as mathematical objects. If the entities of intuitionistic mathematics are recognized as legitimate mathematical objects, intuitionistic mathematics can be accepted as mathematics (as having mathematical content). Let us see, therefore, in which way phenomenology contributes to explaining why intuitionistic entities are indeed mathematical objects.

2. The Phenomenological Justification of Infinitely Proceeding Sequences

Just as in classical mathematics, in intuitionistic mathematics, there are primitive concepts and notions. A case in point is the notion of *infinitely proceeding sequence* of natural numbers (from now on, *ips*) or *choice sequence*. But, while basic concepts (such as *set* and *membership*) in classical mathematics are defined axiomatically, some non-mathematical introduction is felt as needed in intuitionistic mathematics. For example, typically, *ips* are introduced by describing them as entities that are generated by the more or less free unfolding of the idea of *two-ity* (Brouwer 1907: 8). Without entering in any detail,[1] suffice it to say that for Brouwer the idea of *two*-ity—that derives from the observation of the movement of the time as "the falling apart of a life moment into two distinct things"—is the basic *intuition of mathematics*. Making sense of the basic intuition of mathematics by providing a meaningful explanation of what the intuitionist means by *intuition* is, therefore, a necessary step in order to validate the status of *ips* as mathematical objects. According to the supporters of the phenomenological approach to intuitionism, phenomenology has the right conceptual tools for achieving such a goal (van Atten 2003: 4):

> I appeal to phenomenology, as developed by Husserl; who himself, by the way, would have rejected choice sequences, but, it can be argued, unjustly so (van Atten 1999). *It turns out that this phenomenological analysis has a mathematical consequence* (my emphasis): it justifies a particular principle in the theory of choice sequences, the principle of weak continuity (for numbers), for which so far there were only plausibility considerations [...]

The very strong claim (van Atten 2003: 3) is therefore that:

> If you *believe* (my emphasis) Husserl's philosophy of mathematics, then you should also accept Brouwer's choice sequences.

Let us therefore see how Husserl's philosophy of mathematics is applied to intuitionism and in which way it proposes to contribute to promoting the acceptance of infinitely proceeding sequences as mathematical objects.

2.1 Intuiting Mathematical Objects

The main motivation for approaching the intuitionistic philosophical program from a Husserlian phenomenological point of view is that intuitionism and phenomenology share, in a way, the same starting point—namely, that knowledge is a matter of intuition. In particular, they both (are taken to) maintain that "[...] knowledge refers back, directly or indirectly, to intuitions" where intuitions are meant to be "experiences in which objects are actually

1 See, for example, Placek (1999), ch. 2, sec. 2 (*Mathematics and Intuition*), pp. 29–47.

given as themselves" (van Atten 2003: 7). Mathematical knowledge—for example, knowledge of mathematical objects—is taken to be a matter of intuition in the previously specified sense.

This means that in this framework, "intuiting a mathematical object" should be intended as signifying "to have a direct experience of the object". The task that the phenomenologically minded intuitionist philosopher must face is therefore that of providing a philosophical account of the notion of *intuition of a mathematical object* as *direct experience of* a *mathematical object*. The intuitionist phenomenologist does this by describing how intuitions (in general) come about (van Atten 2007: 22):

> Intuitions, to be obtained, generally require a *series of mental acts* (my emphasis). This series of acts has a specific structure that depends on the kind of object to be intuited. The contents of our stream of experiences do not follow one another randomly but are systematically related.

Let us remember that *ips*, denoted α, β, γ, ... are sequences of natural numbers whose values are generated at stages. At any stage t_n, for $n \in \mathrm{N}$, of the construction of a sequence α, a value $\alpha(n)$ of the sequence is introduced together with some restrictions on the future possible values of the sequence. As it appears, the phenomenologist's description of how the act of intuiting happens seems to adapt to the standard description of how infinitely proceeding sequences come into existence.

The main characteristic is that both processes happen at stages as the outcome of a series of choices. In both cases, such choices depend on the previously acquired information. The claim of the phenomenologist is that in both cases, there is a *structure* that can be detected. In one case, it is the structure of a certain type of mathematical entity; in the other case, it is the structure of our experience of such entities. According to the phenomenologist, these two structures match (van Atten 2007: 22):

> There are structures that govern the flow of consciousness. Even what is normally called a 'flash of insight' is a systematic whole or synthesis of acts.

Words like "structure", "govern", and "systematic" are clearly intended to implicitly suggest that *intuition* is not such a nebulous concept—it is about something whose pattern of organization can be detected. Clearly, talking about the "structure" of a series of mental acts seems to be something more manageable than to talk about "intuition", "flow of consciousness", and the like. The phenomenologist's claim is that it amounts to the same thing.

What concerns us here is to see how this type of explanation is meant by the phenomenologist as addressing the main objections to intuitionism—for example, the objection raised by the classical mathematician to the legitimacy of *infinitely proceeding sequences* as mathematical objects. The phenomenologist claims to be able to explain how such entities should be understood so that they become acceptable.

We saw that *ips*, denoted α, β, γ, … are entities (sequences of natural numbers) constructed step-by-step in the course of time t_0, t_1, t_2, ... Given a sequence (or process of construction of a sequence) α, at each stage t_n, the value $\alpha(n)$ is constructed. For the intuitionist, such a construction is unfinished and unfinishable. This means that given any two sequences α and β whose development is exactly the same up to a finite point in time t_m (for $m \in \mathrm{N}$) of our actual construction, one cannot tell that they are indeed the same sequence, as one cannot know whether for all future t_{m+n} for $n, m \in \mathrm{N}$, $n > 0$, we will have $\alpha(m+n) = \beta(m+n)$. One cannot tell, either, that they are different, as this would imply that one is able to indicate a particular $k \in \mathrm{N}$ such that $\alpha(k) \neq \beta(k)$. For the classically minded mathematician, here is a clear difficulty, as she probably would not know how to deal with entities that are essentially undetermined. If we cannot tell whether a process of construction will generate one or more entities, if we cannot tell whether two processes of constructions are identical or apart from each other, we cannot affirm that the generated sequence is indeed an (individual) object. The mathematician, therefore, will probably try to get to know more, looking into principles of reasoning adapted to work with *ips* in a mathematical way.

The philosopher might see things differently. Confronted with the same situation, she will notice that what is missing is a criterion of individuation and will conclude that the possibility of attributing the ontological status of objecthood to *ips* should be figured out in a novel way. This is precisely what phenomenology claims to be able to do—namely, providing a type of criterion of individuation for *ips* that will make them acceptable as mathematical objects. Moving from the idea that intuitionistically there are no non-experienced objects, the phenomenologist will first of all consider the problem of individuation in relation to *our experience of the object* rather than in relation to objects themselves. As a consequence, the phenomenologist will provide an analysis of the *constitution of our experience* in relation to a certain entity, rather than an analysis of the constitution of the entity. The philosophical problem is therefore reformulated, since the question will not be whether *ips* are *objects* but whether *ips* are *objects of our experience*. How does this work?

Since from a phenomenological point of view, objects in general, and mathematical objects in particular, are to be seen as *invariants in our actual experience*, entities of intuitionistic mathematics will be considered as *objects (of our experience*) if we can show that in our experience of them there is such an invariant. The problem is therefore to eventually find the right invariant. This is not an easy task, because—remembering that we are not talking about objects but about our experience (or the experience of an idealized subject) of an object—there is no obvious candidate for what to consider an invariant. The relevant question is: what remains *invariant* in our experience of infinitely proceeding sequences? According to Mark van Atten, what remains invariant is the fact that we experience them as developing sequences whose development started at a particular point in time (van Atten 2003: 12):

> I suggest that what remains invariant is the character of the sequence as a developing sequence, a development that started at a particular point in time.

This is the phenomenological solution to the problem of the legitimation of *ips* as mathematical objects. According to the phenomenologist, the concept of *ips* can be further clarified as follows (van Atten 2003: 13):

> The way in which a choice sequence is an object has much in common with the way another, more familiar type of object is: a melody. An ongoing melody is experienced as an identity even though it has not been completed yet.

This is the type of argument that, according to the phenomenologist, should help in convincing mathematicians that the entities of intuitionistic mathematics are, indeed, legitimate mathematical objects. The phenomenologist goes further than that in claiming that the acceptance of the phenomenological approach is necessary for the right understanding of the intuitionistic point of view. This is the point we will discuss in the next, concluding, section.

3. The Impact of the Phenomenological Approach to Intuitionism on the Intuitionistic Mathematical Program

Summarizing, the claim that intuitionism is first of all a philosophy and that such a philosophy is intended (among other things) to provide a sound foundation to intuitionistic mathematics has as a main consequence that a philosophical approach to intuitionism should be elaborated in a way that is able to answer all the specific objections mathematicians normally raise when confronted with intuitionistic mathematics. In practice, the claim is that to find the answers to the foundational problems raised by intuitionistic mathematics, one should look into philosophy. Clearly, a philosophical defense of intuitionistic mathematics, in order to be effective, should prove itself to have very strong arguments, arguments that are able to counteract mathematical objections. The phenomenologist claims that such arguments can be found applying the phenomenological method.

There is a key fact to keep in mind—namely, that the acceptance of the phenomenological analysis is considered, in this context, an essential step towards the acceptance of intuitionistic mathematics. Since this is a very bold claim, it is important to evaluate what its consequences are. As the role of the philosophical intuitionistic program is to support the intuitionistic mathematical program, the relevant question is whether the phenomenological defense of intuitionism can really promote the acceptance of intuitionistic mathematics as mathematics. We think that in order to answer such a question, we should first ask to whom the phenomenological analysis is directed.

Obviously, the phenomenologist's argument trying to clarify in which sense infinitely proceeding sequences are genuine mathematical objects should be seen as addressing those who do not already accept infinitely proceeding sequences as such. We should then ask whether the phenomenologist's argumentation (outlined in section 2) may possibly be able to convince them. In order to discuss this, we will distinguish two cases: the case of those who refuse *ips* from a mathematical point of view while being philosophically neutral (and therefore possibly open to any philosophical approach) and the case of those who are neutral in respect to what is considered a mathematical object or not (and therefore willing to consider the entities of intuitionistic mathematics as mathematical objects) but are not sympathetic with the phenomenological approach. All the other possible cases are obviously not relevant in this context, as there is no need to convince those who are already convinced, accepting both intuitionism and phenomenology, and there is no way to convince those who strictly refuse both intuitionism and phenomenology.

First case: those who do not accept the notion of a sequence that develops over time as mathematically meaningful while perhaps remaining neutral as to which philosophical approach (if any) to embrace. To our view, those belonging to this class would not consider the type of *invariant* described by the phenomenologist as a palatable substitute for a more standard criterion of individuation. In particular, they would not see that the offered explanation adds new knowledge, new insight, of the type they are looking for. Being told that infinitely proceeding sequences are objects, as they are individuated as evolving processes that started at a particular point in time, they will not be able to find a single reason to accept such entities as the objects of their daily work. When, eventually, the phenomenologist will add, to clarify, that infinitely proceeding sequences are objects more or less in the sense in which a melody is an object, they will just conclude that there really is nothing new to learn.

In our view, a mathematician may well decide to work with mathematical entities, such as infinitely proceeding sequences, even in the case where there is doubt about whether such entities are legitimate mathematical objects or not. The important thing would be to know what to do with them—notably, to be acquainted with the adapted principles of reasoning. In intuitionistic mathematics, there are such principles of reasoning—the so-called *continuity principles*. This is all that the mathematician needs. Perhaps such a mathematician will think that the entities she is using are not really mathematical objects; still, she will perceive her activity as mathematics. This is the type of insight she looks for. On the contrary, the phenomenologist's attempt to justify infinitely proceeding sequences (section 2) will be perceived as being devoid of any mathematical significance. In conclusion, those belonging to this class will be prone to see that phenomenology is just adding a further difficulty. When confronted with the claim that philosophy should be accepted first, they will just raise a very simple objection by remarking that the proposed substitute for a criterion of individuation does not seem to be relevant enough and will reject intuitionism on the basis of its philosophy rather than on the basis of its mathematics.

Second case: those who are not sympathetic with the phenomenological approach while being willing to look into intuitionistic mathematics. Those belonging to this class clearly will not accept the phenomenological shift from "objects" to "experience of objects" as legitimate, no matter what such a shift is supposed to achieve. While for the first case it was a question of content, here, it is a question of methodology. Objects are one thing; experiences are another. Mathematics is not about *experiences*, they will observe. Mathematical conclusions and proofs, for them, are not conclusions and proofs about experiences. While the members of this class are not willing to accept phenomenology, they are open to looking into intuitionistic mathematics. Again, the important thing is to know what to do with the entities of intuitionistic mathematics, how to use them. For this, we need principles of reasoning. They will therefore simply look for such principles and perhaps will end up doing some intuitionistic mathematics. But, if before actually doing intuitionistic mathematics they come to believe that, as the phenomenologist claims, a phenomenological introduction is necessary, they will simply stop in their attempt. Also, in this case, intuitionistic mathematics will be rejected on the basis of its philosophy.

In conclusion, in our view, what the phenomenologist achieves in insisting that phenomenology should be accepted first, that its arguments and methodology should be applied in order to reach a real grasp of the entities of intuitionistic mathematics, is to offer a very easy ground for dismissing intuitionistic mathematics. In both of the cases we described, in fact, the dismissal of infinitely proceeding sequences as genuine mathematical objects, and therefore the dismissal of intuitionistic mathematics as mathematics, will not be based on mathematical reasons.

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