Alethic-Deontic Logic
and the Alethic-Deontic Octagon

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Abstract
This paper will introduce and explore a set of alethic-deontic systems. Alethic-deontic logic is a form of logic that combines ordinary (alethic) modal logic, which deals with modal concepts such as necessity, possibility and impossibility, and deontic logic, which investigates normative expressions such as "ought", "right" and "wrong". I describe all the systems axiomatically. I say something about their properties and prove some theorems in and about them. We will be especially interested in how the different deontic and modal concepts are related to each other in various systems. We will map these relationships in an alethic-deontic octagon, a figure similar to the classical so-called square of opposition.

1. Introduction

In this paper I introduce and explore a set of alethic-deontic systems. Alethic-deontic logic is a kind of bimodal logic that combines ordinary (alethic) modal logic and deontic logic. Introductions to ordinary (alethic) modal logic can be found in e.g. Chellas (1980), Blackburn, de Rijke, & Venema (2001), Blackburn, van Benthem & Wolter (eds.) (2007), Fitting & Mendelsohn (1998), Gabbay (1976), Gabbay & Guenthner (2001), Kracht (1999), Garson (2006), Girle (2000), Lewis & Langford (1932), Popkorn (1994), Segerberg (1971), and Zeman (1973). This branch of logic deals with modal concepts, such as necessity, possibility and impossibility, modal sentences, arguments and systems. Introductions to deontic logic can be found in e.g. Gabbay, Horty, Parent, van der Meyden & van der Torre (eds.) (2013), Hilpinen (1971), (1981), Rönnedal (2010), and Åqvist (1987), (2002). Deontic logic deals with normative words, such as "ought", "right" and "wrong", normative sentences, arguments and systems. For more information about bimodal systems in general and alethic-deontic logics in particular, see e.g. Rönnedal (2012), (2012b), (2015), (2015b). Alethic-deontic logic combines ordinary alethic modal logic and deontic logic. Every axiomatic system in this paper is sound and complete with respect to its semantics (see Rönnedal (2012) and (2012b) for a proof). The present paper includes more information about these systems; I prove several theorems in and about them. We will be especially interested in the relationships between the different modal and normative concepts in various systems. We will use an alethic-deontic octagon, a figure similar to the classical so-called square of opposition, to map these relationships.[1]

The paper is divided into five sections. Section 2 is about syntax and semantics and section 3 about proof theory. Section 4 is the main part of the paper, in which I describe a set of normal alethic-deontic systems. Finally, section 5 includes information about the relationships between the systems I describe.

2. Syntax and semantics

We use the same kind of syntax and semantics as in Rönnedal (2015). However, we introduce a new deontic operator, U (unobligatory), defined in the following way: Up $\leftrightarrow$ $\neg$Op. Furthermore, we use slightly different symbols and treat O and $\Box$ as primitive in this essay; all other operators are defined in terms of O and $\Box$ in a standard way. $\triangledown$p (alethic contingency) = $\neg\Box p \wedge \neg\Box\neg p$; $\triangle$p (alethic non-contingency) = $\Box p \vee \Box\neg p$; $\boxminus$p (unnecessary) = $\neg\Box$p. $\top$ (Verum) = e.g. $p \vee \neg p$, $\bot$ (Falsum) = e.g. $\neg\top$.

Without further ado, let us turn to proof theory.

3. Proof theory

3.1 Systems of alethic-deontic logic

In this paper a system is usually identified with a set of sentences, not a set of theorems together with a deductive apparatus. The concept of a theorem is defined in the standard way (see e.g. Rönnedal (2010)).

Definition 1 (Alethic-deontic system). A set of sentences S is *a system of alethic-deontic logic* or simply an *alethic-deontic logic* or an *alethic-deontic system* (“ad” for short) if and only if:

(i) it contains all propositional tautologies,

[1] Anderson was perhaps the first philosopher to combine alethic and deontic logic (see Anderson (1956)). Fine & Schurz (1996), Gabbay & Guenthner (2001), Gabbay, Kurucz, Wolter & Zakharyaschev (2003), Kracht (1999), and Kracht & Wolter (1991) offer more information about how to combine various logical systems.

(ii) it is closed under modus ponens (MP) (if A is in S and A → B is in S, then so is B), and

(iii) it is closed under uniform substitution (if A belongs to S, then every (immediate) substitution instance of A is in S).

The concept of a substitution instance of A is defined in the usual way (see e.g. Rönnedal (2010)). "PL" (as in "propositional logic") contains every sentence that is valid due to its truth-functional nature. When we are talking about ad systems we presuppose that we are using a language that includes both deontic and alethic operators and not just alethic or just deontic terms. So, PL will include sentences that are not theorems in ordinary propositional logic or in pure deontic or alethic systems. For example PL contains not just ¬(p∧¬p) and p∨¬p, but also for instance ¬(□Op∧¬□Op) and ◇Pp∨¬◇Pp. In a proof, "PL" may also indicate that the step is propositionally correct.

If it is clear from the context that we are speaking of alethic-deontic systems and alethic-deontic logics we will sometimes drop the word "alethic-deontic" and speak only of logics and of systems.

Example 2 (ad systems). (i) The inconsistent system, i.e. the set of *all* sentences is an alethic-deontic logic. This system is the largest alethic-deontic system, since every logic is included in it. (ii) Let **L** be a collection of alethic-deontic systems. Then the intersection of **L** is an alethic-deontic system too, where the intersection of **L** is defined in the standard way. (iii) The logic of any alethic-deontic frame is an alethic-deontic system. (iv) This is also true for logics of classes of alethic-deontic frames. (iv) PL ("propositional logic") is an alethic-deontic system. Since PL is a subset of every alethic-deontic logic, PL is the smallest alethic-deontic system.

We shall say that an *alethic-deontic system* S is *generated by* a set of sentences G iff S is the smallest alethic-deontic logic containing every sentence in G. PL, the set of all "tautologies" is generated by the empty set.

Definition 3 (Normal alethic-deontic system). An alethic-deontic system is *normal* if and only if:

(i) it contains the sentences □(p → q) → (□p → □q), ◇p ↔ ¬□¬p, ◇̶p ↔ □¬p, □̶p ↔ ¬□p, ▽p ↔ (¬□p ∧ ¬□¬p), △p ↔ (□p ∨ □¬p), O(p → q) → (Op → Oq), Pp ↔ ¬O¬p, Fp ↔ O¬p, Up ↔ ¬Op, Kp ↔ (¬Op ∧ ¬O¬p), Np ↔ (Op ∨ O¬p), and

(ii) it is closed under the rules of □-necessitation and O-necessitation (i.e if ⊢ A, then ⊢ □A, and if ⊢ A, then ⊢ OA).

Example 4 (Normal ad systems). (i) The inconsistent system is a normal aletic-deontic logic. (ii) PL is not a normal ad system. However, PL is included in every normal ad system. (iii) Let **L** be a collection of normal alethic-deontic systems. Then the intersection of **L** is a normal alethic-deontic system too. (iv) The logic of any alethic-deontic frame is a normal alethic-deontic system. (v) This is true also for the logic of every class of alethic-deontic frames. (vi) The pure deontic system **dK** (= **OK**) (Rönnedal (2010)) is not a normal alethic-deontic logic. Neither is the pure alethic system **aK** (Chellas (1980)). However, it follows from the definition that every normal ad system includes the minimal normal alethic logic **aK** and the minimal normal deontic logic **dK**.

The smallest normal ad system will be called "minimal alethic-deontic logic" (**MADL**) or **aKdK**.

When we speak of alethic-deontic systems in this essay, it is usually normal alethic-deontic systems we mean.

3.2 Normal alethic-deontic systems

3.2.1 Axioms

A normal alethic-deontic system can be represented by adding axioms to the minimal alethic-deontic logic **MADL**. We will consider three different kinds of axioms in this essay: pure deontic axioms, pure (alethic) modal axioms and bimodal (alethic) modal deontic axioms. And we will use these axioms to construct some normal alethic-deontic systems. The (alethic) modal axioms include aK, aT, aD, a4, aB and a5 (see table 1), well known from ordinary modal logic. The deontic axioms include dK, dD, d4, dT′, dB′ and d5 (see table 2), well know from pure deontic logic. We also consider nine bimodal axioms, i.e. axioms that contain both deontic and (alethic) modal operators, namely, MO, OC, OC′, MO′, ad4, ad5, PMP, OMP, MOP (see table 3). aK and dK are theorems in every normal alethic-deontic system. However, no other axiom is a theorem in **MADL**. Accordingly, we obtain a whole range of normal alethic-deontic systems by adding any subset of these to **MADL**. A system that fuses two monomodal systems, without any bimodal axioms, will be called an alethic-deontic combination (fusion) or ad combination (fusion) for short. See section 3.2.3 below.[2]

All in all we describe 21 different axioms, 19 besides aK and dK. Every ad system we consider will contain aK and dK and zero or more of the other

[2] All systems in this paper are generated from various axioms, rules of inference and the rule of substitution. An alternative is to use axiom schemas and dispense with the substitution rule. Both "methods" generate the same systems.

19 axioms. In fact, we will focus on the 16 systems that can be constructed from the axioms aD, dD, OC and MO. Some of these are deductively equivalent (see section 5).

3.2.1.1 Pure a-axioms

	a-axiom	Corresponding condition on R
aK	$\Box(p\rightarrow q)\rightarrow(\Box p\rightarrow\Box q)$	-
aT	$\Box p\rightarrow p$	$\forall xxRx$
aD	$\Box p\rightarrow\Diamond p$	$\forall x\exists yxRy$
aB	$p\rightarrow\Box\Diamond p$	$\forall x\forall y(xRy\rightarrow yRx)$
a4	$\Box p\rightarrow\Box\Box p$	$\forall x\forall y\forall z((xRy\wedge yRz)\rightarrow xRz)$
a5	$\Diamond p\rightarrow\Box\Diamond p$	$\forall x\forall y\forall z((xRy\wedge xRz)\rightarrow yRz)$

Table 1

3.2.1.2 Pure d-axioms

	d-axiom	Corresponding condition on S
dK	$O(p\rightarrow q)\rightarrow(Op\rightarrow Oq)$	-
dD	$Op\rightarrow Pp$	$\forall x\exists yxSy$
d4	$Op\rightarrow OOp$	$\forall x\forall y\forall z((xSy\wedge ySz)\rightarrow xSz)$
d5	$Pp\rightarrow OPp$	$\forall x\forall y\forall z((xSy\wedge xSz)\rightarrow ySz)$
dT′	$O(Op\rightarrow p)$	$\forall x\forall y(xSy\rightarrow ySy)$
dB′	$O(POp\rightarrow p)$	$\forall x\forall y\forall z((xSy\wedge ySz)\rightarrow zSy)$

Table 2

3.2.1.3 Mixed ad-axioms

	ad-axiom	Corresponding semantic condition
MO	$\Box p\rightarrow Op$	$\forall x\forall y(xSy\rightarrow xRy)$
OC	$Op\rightarrow\Diamond p$	$\forall x\exists y(xSy\wedge xRy)$
OC′	$O(Op\rightarrow\Diamond p)$	$\forall x\forall y(xSy\rightarrow\exists z(yRz\wedge ySz))$
MO′	$O(\Box p\rightarrow Op)$	$\forall x\forall y\forall z((xSy\wedge ySz)\rightarrow yRz)$
ad4	$Op\rightarrow\Box Op$	$\forall x\forall y\forall z((xRy\wedge ySz)\rightarrow xSz)$
ad5	$Pp\rightarrow\Box Pp$	$\forall x\forall y\forall z((xRy\wedge xSz)\rightarrow ySz)$
PMP	$P\Box p\rightarrow\Box Pp$	$\forall x\forall y\forall z((xSy\wedge xRz)\rightarrow\exists w(yRw\wedge zSw))$
OMP	$O\Box p\rightarrow\Box Op$	$\forall x\forall y\forall z((xRy\wedge ySz)\rightarrow\exists w(xSw\wedge wRz))$
MOP	$\Box Op\rightarrow O\Box p$	$\forall x\forall y\forall z((xSy\wedge yRz)\rightarrow\exists w(xRw\wedge wSz))$

Table 3

3.2.2 Axiomatic systems

We are now in a position to say something more systematic about alethic-deontic systems.

We have seen that **MADL** is the smallest normal alethic-deontic logic. This means that **MADL** is included in every other normal ad system. By adding axioms to the axiomatic basis it is possible to extend this logic.

As usual, we shall say that the normal alethic-deontic logic *generated* or *represented* by a set of sentences **Γ** is the smallest normal alethic-deontic logic that includes all sentences in **Γ**. **MADL** is represented by the empty set, extensions of this system by some non-empty set.

Let "**S**" be the name of a normal alethic-deontic system and "**Γ**" the name of a set of axioms. Then **S** + **Γ** is the smallest normal ad system that includes both **S** and every sentence in **Γ**. A special type of ad systems is called ad combinations (fusions). The name of an ad combination will often have the following form: "**aXdY**", where **X** is a set of alethic axioms and **Y** is a set of deontic axioms (see below). More generally, we shall often write **aXdYadZ** for a normal ad system that can be represented by a set **X** of alethic axioms, a set **Y** of deontic axioms and a set **Z** of bimodal axioms (axioms that include both alethic and deontic operators). The ad combination **aXdY** = **aXdYad∅**. Sometimes we will replace **X**, **Y** and **Z** by names of alethic, deontic or alethic-deontic axioms or systems, respectively.

Example 5. a∅d∅ad∅ = **MADL**. Let **X** = {aT}, **Y** = {dD} and **Z** = {MO, OC}. Then **aXdYadZ** = **aTdDadMOOC** = **MADL** + {aT} ∪ {dD} ∪ {MO, OC} = **MADL** + {aT, dD, MO, OC} = **MADL** + {$\Box p \to p$, $Op \to Pp$, $\Box p \to Op$, $Op \to \Diamond p$}. Let **X** = {aT, aB, a4}, **Y** = {} and **Z** = {OC′}. Then **aXdYadZ = aTB4d∅adOC′** = **MADL** + {aT, aB, a4} ∪ {} ∪ {OC′} = **MADL** + {aT, aB, a4, OC′} = **MADL** + {$\Box p \to p$, $p \to \Box\Diamond p$, $\Box p \to \Box\Box p$, $O(Op \to \Diamond p)$}. **a∅dSDLad∅** = **MADL** + {dD} = **MADL** + {$Op \to Pp$}. **aS5dOS5+adMOOC** = **MADL** + {aT, aD, aB, a4, a5, dD, dT′, dB′, d4, d5, MO, OC}. Since aK and dK are included in every normal ad system, it is not necessary to mention them in the name of a system. E.g. the following identities hold: **aKdKad∅** = **a∅d∅ad∅, aKTdK5adOC** = **aTd5adOC** and **aK45dKad∅** = **a45d∅ad∅.**

3.2.3 ad combinations (fusions)

Let us say something more about ad combinations (fusions). A (normal) alethic-deontic system **adS** is called the combination (fusion) of a (normal) alethic modal system **aS** and a (normal) deontic system **dS**, written **aS**+**dS**, if and only if **adS** is the smallest (normal) ad system that includes both **aS** and **dS**. If **aS** is representable as **aX** and **dS** as **dY**, then **aS** + **dS** is representable as **aXdY**, where **X** and **Y** is a set of alethic and a set of deontic axioms, respectively. Hence, **aX** + **dY** = **aXdY**.

Note that **aXdY** ≠ **aX** ∪ **dY**, i.e. the ad combination of **aX** and **dY** is not identical to the union of the pure alethic system **aX** and the pure deontic system **dY**. For **aXdY** contains sentences that are not included in **aX** ∪ **dY**. Every normal ad system contains O-nec and □-nec. So, both □O(p → p) and O□(p → p) are, for instance, elements in **aXdY**, but not in **aX** ∪ **dY**. Other examples are the following sentences: □O(p → q) → (□Op → □Oq), ◇O(p ∧ q) ↔ ◇(Op ∧ Oq) and (□Fr ∧ □O((p ∨ q) → r)) → (□Fp ∧ □Fq). Furthermore, additional axioms together with one or more rules of inference may generate sentences that are theorems of the combination of **aS** and **dS** that are not theorems in the union of **aS** and **dS**. E.g. suppose that □p → p ∈ **X**, then □Pp → Pq ∈ **aXdY** but not □Pp → Pq ∈ **aX** ∪ **dY** (since □Pp → Pq is neither an element in **aX** nor in **dY**), or that Op → Pp ∈ **Y**, then □Op → □Pp ∈ **aXdY**, but not □Op → □Pp ∈ **aX** ∪ **dY** (since □Op → □Pp is neither an element in **aX** nor in **dY**). However, the union of the pure alethic system **aX** and the pure deontic system **dY** is of course a subset of the combination of **aX** and **dY**, **aX** ∪ **dY** ⊆ **aXdY**, i.e. everything included in **aX** ∪ **dY** is also included in **aXdY**. It follows that **aX** ∪ **dY** ⊂ **aXdY**.

4. Some normal alethic-deontic systems

I will now consider some normal alethic-deontic systems and I will prove some theorems in and about these systems.

4.1 Minimal alethic-deontic logic

Minimal alethic-deontic logic (**MADL**, **aKdK**, **aKdKad∅** or **a∅d∅ad∅**) is the smallest normal alethic-deontic logic. We will also call this system **S1**. Since it is a *normal* alethic-deontic system **MADL** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. Since it is the *smallest* normal alethic-deontic logic it contains no other axioms or rules of inference. A normal **aKdKad∅**-system is any normal alethic-deontic extension of **aKdKad∅**, i.e. every normal alethic-deontic system is a normal **aKdKad∅**-system, or, in other words, every normal ad system is an extension of **MADL**. This is true by definition and trivial.

MADL is an ad combination of the purely deontic system **dK** (= **OK)** and the purely alethic system **aK**. Hence, we can also call this system **aKdK** or simply **a∅d∅**. Recall that an ad combination of two systems is not the same as the union of these systems (section 3.2.3). So, **aKdK** ≠ **aK** ∪ **dK**. **aKdK** has theorems that are not elements in **aK** ∪ **dK** (e.g. $\Box O(p \to q) \to \Box(Pp \to Pq)$). On the other hand, every sentence that belongs to either **aK** or **dK** is an element in **aKdK,** i.e. if s ∈ **aK** ∪ **dK**, then s ∈ **aKdK**, for any sentence s. It follows that if any formula is a theorem in either **aK** or **dK** it is also a theorem in every normal alethic-deontic logic.

I will now prove some theorems in and about **MADL**. Since **MADL** is included in every normal ad logic, these theorems hold in every ad system we consider in this essay.

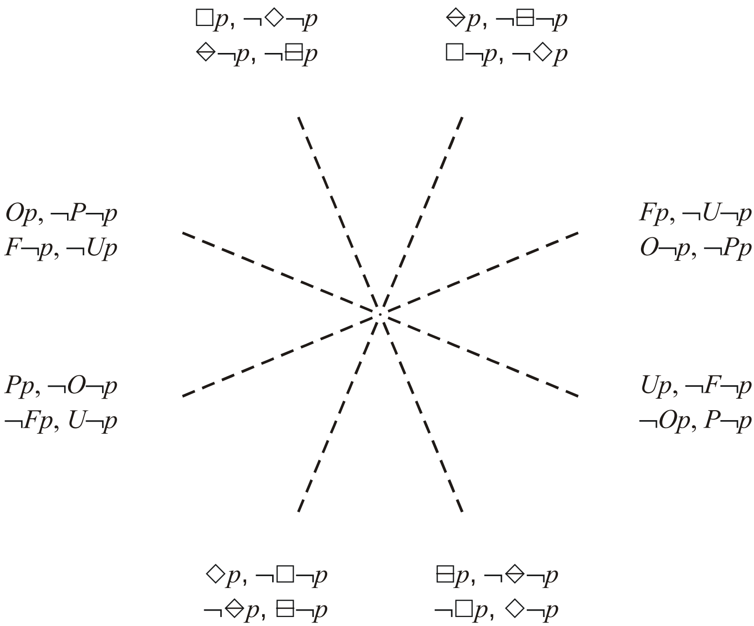

Figure 1. The Alethic-Deontic Octagon, MADL (S1).

4.1.1 The alethic-deontic octagon

It is possible to display some important logical relationships between O-, P-, F- and U-sentences in various deontic systems in a deontic square of opposition (see Rönnedal (2010)) and some important logical relationships between □-, ◇-, ⟠- and ⊟-sentences in various alethic systems in a similar alethic square of opposition. If we combine these figures we get something I will call the alethic-deontic octagon. This is a figure that can be used to represent some of the most important logical relationships between all primary deontic and alethic sentences, i.e. all of the following formulas: Op, Pp, Fp, Up, □p, ◇p, ⟠p and ⊟p. These relationships will vary from one ad system to another.

Figure 1 shows what the ad octagon looks like in **MADL**. All sentences that occur at a "node" in the figure are equivalent (e.g. Op↔F¬p and ◇¬p ↔ ¬□p). Sentences that are connected via dashed lines are contradictories (e.g. ¬(□p↔◇¬p), □p↔¬◇¬p, ¬(Pp↔Fp) and Pp↔¬Fp are theorems). Since **MADL** is the smallest ad system, these relationships hold in every ad system. However, all of the relationships displayed in this figure also hold in the union of **aK** and **dK**. So, the figure is perhaps more important for what it does not, than for what it does contain. The ad octagon will become more interesting in extensions of **MADL**.

4.1.2 The rule of replacement

The rule of replacement and the rule of simultaneous replacement hold in every normal ad system. The following section proves this. In our proofs we use the following derived rules: (OEQ) If ⊢ A↔B, then ⊢ OA↔OB, and (□EQ) If ⊢ A↔B, then ⊢ □A↔□B (see Rönnedal (2010) for a proof of the first, the second can be established in a similar way). These rules are derivable in every normal ad system.

The rule of replacement (Rep). (i) If ⊢ A ↔ B, then ⊢ C ↔ [B//A](C) (if A is equivalent to B is a theorem, then C is equivalent to [B//A](C) is a theorem), where [B//A](C) is like C except that zero or more occurrences of A are replaced by B (see Rönnedal (2010) for more information about the concept of replacement).

(ii) If ⊢ A↔B and ⊢ C, then ⊢ [B//A](C) (if A is equivalent to B is a theorem and C is a theorem, then [B//A](C) is a theorem), where C and [B//A](C) are as in part (i).

(iii) If ⊢ A↔B and ⊢ [B//A](C), then ⊢ C (if A is equivalent to B is a theorem and [B//A](C) is a theorem, then C is a theorem), where C and [B//A](C) are as in part (i).

Proof: **Part (i)**. Suppose that the replacement of B for A is at zero places. Then [B//A](C) and C are identical and the result is trivial (⊢ C ↔

[B//A](C), where [B//A](C) = C). Suppose that A and C is the same sentence and that C i.e. A is replaced by B. Then [B//A](C) is B. Hence (i) holds in this case too ($\vdash$ C $\leftrightarrow$ [B//A](C), where C = A and [B//A](C) = B). So, from now on we assume that A and C are distinct and that at least one occurrence of A is replaced by B in C. The rest of the proof is by induction on the length of A. Given $\vdash$ A $\leftrightarrow$ B.

Basis: C is atomic. Since C and A are distinct and C is atomic [B//A](C) = C. Hence, $\vdash$ C $\leftrightarrow$ [B//A](C), where [B//A](C) = C. Consequently the theorem holds when C is atomic.

Induction step. We want to show that if it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C) for C of any complexity, then it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ f(C) $\leftrightarrow$ f([B//A](C)), where f(C) is ¬C, D $\wedge$ C, C $\wedge$ D, D $\vee$ C, C $\vee$ D, D $\rightarrow$ C, C $\rightarrow$ D, D $\leftrightarrow$ C, C $\leftrightarrow$ D, OC, PC, FC, UC, KC, NC, □C, ◇C, ⬖C, ⊟C, ▽C or △C, and likewise for [B//A](C). Since conjunction, disjunction and equivalence are commutative, since equivalence, implication and conjunction can be expressed in terms of negation and disjunction, since P, F, U, K and N are definable in terms of O, and since ◇C, ⬖C, ⊟C, ▽C and △C are definable in terms of □, it is sufficient to consider four cases.

Case (i). ¬C. Suppose that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C). From the hypothesis $\vdash$ A $\leftrightarrow$ B. Hence, $\vdash$ C $\leftrightarrow$ [B//A](C). (C $\leftrightarrow$ [B//A](C)) $\leftrightarrow$ (¬C $\leftrightarrow$ ¬[B//A](C)) is a tautology. Accordingly, $\vdash$ ¬C $\leftrightarrow$ ¬[B//A](C) by PL. It follows that if it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C), then it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ ¬C $\leftrightarrow$ ¬[B//A](C).

Case (ii). C $\vee$ D. Assume that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C). By the hypothesis $\vdash$ A $\leftrightarrow$ B. Thus, $\vdash$ C $\leftrightarrow$ [B//A](C). (C $\leftrightarrow$ [B//A](C)) $\leftrightarrow$ ((C $\vee$ D) $\leftrightarrow$ ([B//A](C) $\vee$ D)) is logically true in propositional logic. Hence, $\vdash$ (C $\vee$ D) $\leftrightarrow$ ([B//A](C) $\vee$ D) by PL. Consequently, if it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C), then it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ (C $\vee$ D) $\leftrightarrow$ ([B//A](C) $\vee$ D).

Case (iii). OC. Suppose that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C). By the hypothesis $\vdash$ A $\leftrightarrow$ B. Hence, $\vdash$ C $\leftrightarrow$ [B//A](C) and so, $\vdash$ OC $\leftrightarrow$ O[B//A](C) by (OEQ). In consequence, if it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ C $\leftrightarrow$ [B//A](C), then it is the case that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ OC $\leftrightarrow$ O[B//A](C).

Case (iv). □C. As in case (iii).

Conclusion. We have now shown that the rule of replacement holds where there are no connectives or operators outside A and B and that if it holds where there are n such logical connectives or operators it holds for n + 1. We conclude that the theorem holds in general.

Part (ii). Assume that (1) $\vdash A \leftrightarrow B$ and $\vdash C$ (A is equivalent to B is a theorem and C is a theorem) and that C and [B//A](C) are as in part (i). Then both (2) $\vdash A \leftrightarrow B$ and (3) $\vdash C$ [from (1)]. From (2) we obtain (4) $\vdash C \leftrightarrow$ [B//A](C) [by PL and part (i)]. Hence, (5) $\vdash$ [B//A](C) [from 3 and 4 by PL]. Consequently, (6) if $\vdash A \leftrightarrow B$ and $\vdash C$, then $\vdash$ [B//A](C) (if A is equivalent to B is a theorem and C is a theorem, then [B//A](C) is a theorem), where C and [B//A](C) are as in part (i) [from 1–5 by conditional proof discharging the assumption].

Part (iii). As in part (ii). Details are left to the reader. ■

The rule of simultaneous replacement. (i) If $\vdash A_1 \leftrightarrow B_1$ and … and $\vdash A_n \leftrightarrow B_n$ then $\vdash C \leftrightarrow [B_1//A_1, …, B_n//A_n](C)$ (if A_1 is equivalent to B_1 and … and A_n is equivalent to B_n are theorems, then C is equivalent to $[B_1//A_1, …, B_n//A_n](C)$ is a theorem), where $[B_1//A_1, …, B_n//A_n](C)$ is the result of replacing zero or more occurrences of A_1 in C by B_1 and … and replacing zero or more occurrences of A_n in C by B_n.

(ii) If $\vdash A_1 \leftrightarrow B_1$ and … and $\vdash A_n \leftrightarrow B_n$ and $\vdash C$, then $\vdash [B_1//A_1, …, B_n//A_n](C)$ (if A_1 is equivalent to B_1 and … and A_n is equivalent to B_n are theorems and C is a theorem, then $[B_1//A_1, …, B_n//A_n](C)$ is a theorem), where C and $[B_1//A_1, …, B_n//A_n](C)$ are as in part (i).

(iii) If $\vdash A_1 \leftrightarrow B_1$ and … and $\vdash A_n \leftrightarrow B_n$ and $\vdash C \leftrightarrow [B_1//A_1, …, B_n//A_n](C)$, then $\vdash C$ (if A_1 is equivalent to B_1 and … and A_n is equivalent to B_n are theorems and $[B_1//A_1, …, B_n//A_n](C)$ is a theorem, then C is a theorem), where C and $[B_1//A_1, …, B_n//A_n](C)$ are as in part (i).

Proof. The proof is more or less obvious, simply use the rule of replacement repeatedly in crucial steps. ■

4.1.3 Interchange and duality theorems

Let us prove some interchange and duality theorems that can be used to quickly prove and recognize new theorems in **MADL** and other ad systems.

Theorem 6 (The ad interchange theorem (adIT)). Let $\otimes_1 \ldots \otimes_n$ be a sequence of deontic and alethic operators in a sentence such that each $\otimes_i$ is O, P, F, U, □, ◇, ⟠ or ⊟. If $\otimes_i$ = O, let $\otimes_i$’ = P, and vice versa, if ⊗ = F, let ⊗’ = U and vice versa, if ⊗ = □, let ⊗’ = ◇ and vice versa and if ⊗ = ⟠, let ⊗’ = ⊟ and vice versa, for every $\otimes_i$. Then, (i) $\vdash \otimes_1 \ldots \otimes_n A \leftrightarrow \neg\otimes_1' \ldots \otimes_n'\neg A$, (ii) $\vdash \neg\otimes_1 \ldots \otimes_n A \leftrightarrow \otimes_1' \ldots \otimes_n'\neg A$, and (iii) $\vdash \otimes_1 \ldots \otimes_n\neg A \leftrightarrow \neg\otimes_1' \ldots \otimes_n' A$, for any A.

Proof. **Part (i).** Let $\otimes_1 \ldots \otimes_n$ be a sequence of deontic and alethic operators of the kind mentioned in the theorem. By PL $\vdash \otimes_1 \ldots \otimes_n A \leftrightarrow \otimes_1 \ldots \otimes_n A$, for any A. Now replace O by $\neg P \neg$, P by $\neg O \neg$, F by $\neg U \neg$, U by $\neg F \neg$, $\Box$ by $\neg \Diamond \neg$, $\Diamond$ by $\neg \Box \neg$, $\diamondminus$ by $\neg \boxminus \neg$ and $\boxminus$ by $\neg \diamondminus \neg$ throughout in the right hand side of this equivalence. Then we get the following theorem $\vdash \otimes_1 \ldots \otimes_n A \leftrightarrow \neg \otimes_1' \neg \neg \otimes_2' \neg \ldots \neg \otimes_{n-1}' \neg \neg \otimes_n' \neg A$, for any A. Use PL and replacement to delete all double negations. It follows that $\vdash \otimes_1 \ldots \otimes_n A \leftrightarrow \neg \otimes_1' \ldots \otimes_n' \neg A$, for any A. This proves part (i).

Part (ii). PL and part (i) gives us $\vdash \neg \otimes_1 \ldots \otimes_n A \leftrightarrow \neg \neg \otimes_1' \ldots \otimes_n' \neg A$, which again by PL immediately proves that $\vdash \neg \otimes_1 \ldots \otimes_n A \leftrightarrow \otimes_1' \ldots \otimes_n' \neg A$, for any A.

Part (iii). By replacing A by $\neg A$ in part (i) we obtain $\vdash \otimes_1 \ldots \otimes_n \neg A \leftrightarrow \neg \otimes_1' \ldots \otimes_n' \neg \neg A$. By PL and replacement it follows that $\vdash \otimes_1 \ldots \otimes_n \neg A \leftrightarrow \neg \otimes_1' \ldots \otimes_n' A$, for any A. ■

Example 7. When n = 0 part (i) reduces to $\vdash A \leftrightarrow \neg \neg A$, part (ii) to $\vdash \neg A \leftrightarrow \neg A$ and part (iii) to $\vdash \neg A \leftrightarrow \neg A$. Let n = 1. Then the following schemas are examples of instances of the theorem $\vdash OA \leftrightarrow \neg P \neg A$, $\vdash PA \leftrightarrow \neg O \neg A$, $\vdash \Box A \leftrightarrow \neg \Diamond \neg A$, $\vdash \Diamond A \leftrightarrow \neg \Box \neg A$ [part (i)], $\vdash \neg OA \leftrightarrow P \neg A$, $\vdash \neg PA \leftrightarrow O \neg A$, $\vdash \neg \boxminus A \leftrightarrow \diamondminus \neg A$, $\vdash \neg \diamondminus A \leftrightarrow \boxminus \neg A$ [part (ii)], and $\vdash O \neg A \leftrightarrow \neg PA$, $\vdash P \neg A \leftrightarrow \neg OA$ [part (iii)]. In fact, many equivalences in figure 1 can be seen as special cases of adIT. Here are some more complex examples: $\vdash \Box O(p \to p) \leftrightarrow \neg \Diamond P \neg (p \to p)$, $\vdash \neg \Box P(p \wedge q) \leftrightarrow \Diamond O \neg (p \wedge q)$, $\vdash \boxminus \diamondminus F \neg (p \to (Pq \wedge Pr)) \leftrightarrow \neg \diamondminus \boxminus U(p \to (Pq \wedge Pr))$.

Theorem 8 (The ad interchange rule (adIR)). All of the following rules are derived in **MADL**. Let $\otimes$ and $\otimes'$ be as in adIT. Then, (i) $\vdash \otimes_1 \ldots \otimes_n A$ iff $\vdash \neg \otimes_1' \ldots \otimes_n' \neg A$, (ii) $\vdash \neg \otimes_1 \ldots \otimes_n A$ iff $\vdash \otimes_1' \ldots \otimes_n' \neg A$, and (iii) $\vdash \otimes_1 \ldots \otimes_n \neg A$ iff $\vdash \neg \otimes_1' \ldots \otimes_n' A$, for any A.

Proof. The proofs are easy and are left to the reader (use the interchange theorem). ■

We can in fact prove something slightly stronger. The interchange theorem does not hold for modalities that include embedded negation signs. But our next theorem does.

Before turning to the duality theorem, we must first introduce the concept of duality.

Definition 9 (Duality). (i) Let L be a language that contains $\neg$, $\wedge$ and $\vee$ as the only propositional connectives. In addition, let L contain Verum and Falsum, and all normal alethic and deontic operators, i.e. O, P, F, U, $\Box$, $\Diamond$, $\diamondminus$, and $\boxminus$. Then *the dual of a sentence A* (in L), in symbols d(A), is defined as the result of replacing every atomic sentence by its negation and interchanging all occurrences of Verum and Falsum, $\wedge$ and $\vee$, $\Box$ and $\Diamond$, O and P, $\diamondminus$ and $\boxminus$, and F and U in A. (ii) Let L by a language that contains $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, Verum, Falsum and every normal alethic and deontic operator. Then if A is a sentence (in L), then *the dual of A*, in symbols d(A), is defined in the following manner.

1 $d(p) = \neg p$, when p is atomic	6 $d(A \vee B) = (d(A) \wedge d(B))$	11 $d(OA) = Pd(A)$
2 d(Verum) = Falsum	7 $d(A \rightarrow B) = (\neg d(A) \wedge d(B))$	12 $d(PA) = Od(A)$
3 d(Falsum) = Verum	8 $d(A \leftrightarrow B) = (d(A) \leftrightarrow \neg d(B))$	13 $d(\diamondminus A) = \boxminus d(A)$
4 $d(\neg A) = \neg d(A)$	9 $d(\Box A) = \Diamond d(A)$	14 $d(\boxminus A) = \diamondminus d(A)$
5 $d(A \wedge B) = (d(A) \vee d(B))$	10 $d(\Diamond A) = \Box d(A)$	15 $d(FA) = Ud(A)$
		16 $d(UA) = Fd(A)$

Example 10. (i) $d(O\top) = P\bot$, (ii) $d(p \wedge q) = (\neg p \vee \neg q)$ (iii) $d(Op \rightarrow Pp) = (\neg P\neg p \wedge O\neg p)$, (iv) $d(Op \rightarrow \Diamond p) = (\neg P\neg p \wedge \Box\neg p)$, (v) $d(Pp \leftrightarrow \neg Fp) = (O\neg p \leftrightarrow \neg\neg U\neg p)$, (vi) $d(\Box(p \rightarrow q) \rightarrow (Op \rightarrow Oq)) = (\neg\Diamond(\neg\neg p \wedge \neg q) \wedge (\neg P\neg p \wedge P\neg q))$, (vii) $d(P(p \wedge q) \rightarrow \Diamond(p \vee q)) = (\neg O(\neg p \vee \neg q) \wedge \Box(\neg p \wedge \neg q))$, (viii) $d(((Fq \wedge Fr) \wedge \Box(p \rightarrow (q \vee r))) \rightarrow Fp) = (\neg((U\neg q \vee U\neg r) \vee \Diamond(\neg\neg p \wedge (\neg q \wedge \neg r))) \wedge U\neg p)$, (ix) $d(Up \rightarrow \boxminus p) = (\neg F\neg p \wedge \diamondminus\neg p)$, (x) $d((O(p \vee q) \wedge \diamondminus p) \rightarrow Pq) = (\neg(P(\neg p \wedge \neg q) \vee \boxminus\neg p) \wedge O\neg q)$.

Proof. We prove part (vi) and (vii) and leave the rest to the reader.

Part (vi). $d(\Box(p \rightarrow q) \rightarrow (Op \rightarrow Oq)) = (\neg d(\Box(p \rightarrow q)) \wedge d(Op \rightarrow Oq))$ [part 7] $= (\neg\Diamond d(p \rightarrow q) \wedge d(Op \rightarrow Oq))$ [part 9] $= (\neg\Diamond(\neg d(p) \wedge d(q)) \wedge d(Op \rightarrow Oq))$ [part 7] $= (\neg\Diamond(\neg d(p) \wedge d(q)) \wedge (\neg d(Op) \wedge d(Oq)))$ [part 7] $= (\neg\Diamond(\neg d(p) \wedge d(q)) \wedge (\neg Pd(p) \wedge Pd(q)))$ [part 11] $= (\neg\Diamond(\neg\neg p \wedge \neg q) \wedge (\neg P\neg p \wedge P\neg q))$ [part 1].

Part (vii). $d(P(p \wedge q) \rightarrow \Diamond(p \vee q)) = (\neg d(P(p \wedge q)) \wedge d(\Diamond(p \vee q)))$ [part 7] $= (\neg Od(p \wedge q) \wedge d(\Diamond(p \vee q)))$ [part 12] $= (\neg O(d(p) \vee d(q)) \wedge d(\Diamond(p \vee q)))$ [part 5] $= (\neg O(d(p) \vee d(q)) \wedge \Box d(p \vee q))$ [part 10] $= (\neg O(d(p) \vee d(q)) \wedge \Box(d(p) \wedge d(q))$ [part 6] $= (\neg O(\neg p \vee \neg q) \wedge \Box(\neg p \wedge \neg q))$ [part 1]. ■

Theorem 11 (The duality theorem (DUAL)). Let **S** be a normal alethic-deontic system. Then **S** has the following theorems and rules of inference.

(i) $\vdash_S A \leftrightarrow \neg d(A)$.
(ii) $\vdash_S \neg A \leftrightarrow d(A)$.
(iii) if $\vdash_S A$, then $\vdash_S \neg d(A)$.
(iv) if $\vdash_S \neg A$, then $\vdash_S d(A)$.
(v) if $\vdash_S A \rightarrow B$, then $\vdash_S d(B) \rightarrow d(A)$.
(vi) if $\vdash_S A \leftrightarrow B$, then $\vdash_S d(A) \leftrightarrow d(B)$.

Proof. Assume throughout that **S** is a normal alethic-deontic system.

Part (i) $\vdash_S$ A ↔ ¬d(A). Part (i) says that A and the negation of the duality of A are equivalent in S. We want to show that the theorem holds for any sentence regardless of complexity. We prove this by induction on the lenght of A.

Basis. A is atomic. (1) ⊢ p ↔ ¬¬p, for every atomic sentence p [by PL]. Hence, (2) ⊢ p ↔ ¬d(p), for every atomic sentence p [from 1 and the definition of duality part 1]. Consequently, the theorem holds when A is atomic.

Induction step. We want to show that if the theorem holds for a sentence A of given complexity, it holds for every sentence of next higher degree of complexity. Induction hypotheses: the theorem holds for every sentence B and C shorter than A, i.e. ⊢ B ↔ ¬d(B) and ⊢ C ↔ ¬d(C). A = B ∧ C, A = B → C, A = B ↔ C, A = ⊟B, and A = UB. Left as exercise.

A = ¬B. ⊢ ¬B ↔ ¬¬d(B) [by the induction hypothesis and PL]. Consequently, ⊢ ¬B ↔ ¬d(¬B) [by the definition of duality part 4].

A = B ∨ C. (1) ⊢ (B ∨ C) ↔ (¬d(B) ∨ ¬d(C)) [induction hypothesis and replacement]. (2) ⊢ (¬d(B) ∨ ¬d(C)) ↔ ¬(d(B) ∧ d(C)) [by PL]. (3) ⊢ ¬(d(B) ∧ d(C)) ↔ ¬d(B ∨ C) [by the definition of duality part 6 and replacement]. (4) ⊢ (B ∨ C) ↔ ¬d(B ∨ C) [from 1, 2 and 3 by PL].

A = □B. This is exactly as in the case A = OB (see below), just replace every occurrence of O by □ throughout and replace the justification for step (3) by “the definition of duality part 9”.

A = ◇B. (1) ⊢ ◇B ↔ ◇¬d(B) [induction hypothesis, replacement]. (2) ⊢ ◇¬d(B) ↔ ¬□d(B) [definition of ◇ and replacement]. (3) ⊢ ¬□d(B) ↔ ¬d(◇B) [by the definition of duality part 10]. It follows that ⊢ ◇B ↔ ¬d(◇B) [from 1, 2 and 3 by PL].

A = ⟠B. Similar to the case where A = FB (see below).

A = OB. (1) ⊢OB ↔ O¬d(B) [induction hypothesis, replacement]. (2) ⊢ O¬d(B) ↔ ¬Pd(B) [definition of P, PL]. (3) ⊢ ¬Pd(B) ↔ ¬d(OB) [by the definition of duality part 11]. Thus, (4) ⊢ OB ↔ ¬d(OB) [from 1, 2 and 3 by PL].

A = PB. This is exactly as in the case A = ◇B (see above), just replace every occurrence of ◇ by P throughout and replace the justification for step (3) by “the definition of duality part 9”.

A = FB. (1) ⊢FB ↔ F¬d(B) [induction hypothesis, replacement]. (2) ⊢ F¬d(B) ↔ ¬Ud(B) [interchange]. (3) ⊢ ¬Ud(B) ↔ ¬d(FB) [the definition of duality part 15]. Thus, (4) $\vdash$ FB $\leftrightarrow$ ¬d(FB) [from 1, 2 and 3 by PL].

Part (ii) $\vdash_S$ ¬A $\leftrightarrow$ d(A). Part (ii) follows immediately from part (i) by PL. The interpretation is similar.

Part (iii) if $\vdash_S$ A, then $\vdash_S$ ¬d(A). According to part (iii) the negation of the duality of A is a theorem in S, if A is a theorem in S. Suppose that $\vdash$ A. Then by part (i) and (MP) it follows that $\vdash$ ¬d(A). So, if $\vdash$ A, then $\vdash$ ¬d(A) [by conditional proof].

Part (iv) if $\vdash_S$ ¬A, then $\vdash_S$ d(A). This part is proved as part (iii), but use part (ii) instead of part (i) in the proof. It is interpreted similarly.

Part (v) if $\vdash_S$ A $\rightarrow$ B, then $\vdash_S$ d(B) $\rightarrow$ d(A). Part (v) says that if A implies B is a theorem, then the duality of B implies the duality of A is a theorem. Suppose (1) $\vdash$ A $\rightarrow$ B. Then by part (i) and replacement we get (2) $\vdash$ ¬d(A) $\rightarrow$ ¬d(B) [from 1]. Accordingly, (3) $\vdash$ d(B) $\rightarrow$ d(A) [from 2 by PL]. It follows that if $\vdash$ A $\rightarrow$ B, then $\vdash$ d(B) $\rightarrow$ d(A) [by conditional proof from 1 – 3].

Part (vi) if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ d(A) $\leftrightarrow$ d(B). If it is a theorem that A is equivalent to B, then it is a theorem that the duality of A is equivalent to the duality of B, according to this part. Suppose $\vdash$ A $\leftrightarrow$ B. Then (2) $\vdash$ A $\rightarrow$ B and (3) $\vdash$ B $\rightarrow$ A [from 1 by PL]. (4) $\vdash$ d(B) $\rightarrow$ d(A) [from 2 and part (v)]. (5) $\vdash$ d(A) $\rightarrow$ d(B) [from 3 and part (v)]. Hence, (6) $\vdash$ d(A) $\leftrightarrow$ d(B) [from 4 and 5 by PL]. It follows that if $\vdash$ A $\leftrightarrow$ B, then $\vdash$ d(A) $\leftrightarrow$ d(B) [by conditional proof from 1 – 5]. ■

Theorem 12 (The duality corollary (Dual)). The *dual of an alethic-deontic modality* M, D(M), is the modality that is obtained from M by interchanging O and P, F and U, □ and ◇, and ⊟ and ⟠, respectively, throughout. Let M and N be alethic-deontic OPFU□◇⊟⟠ modalities and D(M) and D(N) be the dual of M and N respectively. Then:

Part (i) $\vdash$ MA$\leftrightarrow$¬D(M)¬A.
Part (ii) $\vdash$ MA iff $\vdash$ ¬D(M)¬A.
Part (iii) $\vdash$ MA$\rightarrow$NA iff $\vdash$ D(N)A$\rightarrow$D(M)A.
Part (iv) $\vdash$ MA$\leftrightarrow$NA iff $\vdash$ D(M)A$\leftrightarrow$D(N)A.

Proof. **Part (i).** $\vdash$ MA$\leftrightarrow$¬D(M)¬A. (1) $\vdash$ MA$\leftrightarrow$¬d(MA) [Dual]. (2) $\vdash$ ¬d(MA)$\leftrightarrow$¬D(M)dA [PL, the definition of duality]. (3) $\vdash$ ¬D(M)dA $\leftrightarrow$¬D(M)¬A [PL, Dual, replacement]. (4) $\vdash$ MA$\leftrightarrow$¬D(M)¬A [from 1, 2 and 3 by PL].

Part (ii). (1) $\vdash$ MA $\rightarrow$¬D(M)¬A [from (i) and PL]. Suppose (2) $\vdash$ MA. Then (3) $\vdash$ ¬D(M)¬A [from 1 and 2 by (MP)]. Hence, (4) if $\vdash$ MA, then $\vdash \neg D(M)\neg A$ [from 2 – 3 by conditional proof]. (5) $\vdash \neg D(M)\neg A \rightarrow MA$ [from (i) and PL]. Suppose (6) $\vdash \neg D(M)\neg A$. Then (7) $\vdash MA$ [from 5 and 6 by (MP)]. So, (8) if $\vdash \neg D(M)\neg A$, then $\vdash MA$ [from 6 – 7 by conditional proof]. It follows that $\vdash MA$ iff $\vdash \neg D(M)\neg A$ [from 4 and 8 by classical logic].

Part (iii). (1) $\vdash MA \rightarrow NA$ iff (2) $\vdash \neg D(M)\neg A \rightarrow \neg D(N)\neg A$ [from 1 by replacement] iff (3) $\vdash D(N)\neg A \rightarrow D(M)\neg A$ [from 2 by PL] iff (4) $\vdash D(N)A \rightarrow D(M)A$ [from 3 by PL and replacement]; in conclusion, (5) $\vdash MA \rightarrow NA$ iff $\vdash D(N)A \rightarrow D(M)A$ [from 1 – 4 by classical logic].

Part (iv). Suppose (1) $\vdash MA \leftrightarrow NA$. Then (2) $\vdash MA \rightarrow NA$ [from 1 by PL] and (3) $\vdash NA \rightarrow MA$ [from 1 by PL]. (4) $\vdash D(N)A \rightarrow D(M)A$ [from 2 and part (iii)]. (5) $\vdash D(M)A \rightarrow D(N)A$ [from 3 and part (iii)]. (6) $\vdash D(M)A \leftrightarrow D(N)A$ [from 4 and 5 by PL]. Consequently, (7) if $\vdash MA \leftrightarrow NA$ then $\vdash D(M)A \leftrightarrow D(N)A$ [by conditional proof from 1 – 6]. Suppose (8) $\vdash D(M)A \leftrightarrow D(N)A$. Then (9) $\vdash D(M)A \rightarrow D(N)A$ [from 8 by PL] and (10) $\vdash D(N)A \rightarrow D(M)A$ [from 8 by PL]. (11) $\vdash NA \rightarrow MA$ [from 9 and part (iii)]. (12) $\vdash MA \rightarrow NA$ [from 10 and part (iii)]. (13) $\vdash MA \leftrightarrow NA$ [11, 12, PL]. Consequently, (14) if $\vdash D(M)A \leftrightarrow D(N)A$, then $\vdash MA \leftrightarrow NA$ [from 8 – 13 by conditional proof]. It follows that (15) $\vdash MA \leftrightarrow NA$ iff $\vdash D(M)A \leftrightarrow D(N)A$ [from 7 and 14 by classical logic]. ■

Comment 13. Note that both the duality theorem and the duality corollary are abbreviated "Dual". When any theorem or any rule that is part of one of these propositions is used, we will indicate this by writing "Dual" in the justificatory entry.

The following theorem illustrates how the duality corollary can be used.

Theorem	**"Dual" theorem**	**Theorem**	**"Dual" theorem**
1 $Op \rightarrow OOp$	T(1) $PPp \rightarrow Pp$	7 $\Box p \rightarrow Op$	T(7) $Pp \rightarrow \Diamond p$
2 $Pp \rightarrow OPp$	T(2) $POp \rightarrow Op$	8 $Op \rightarrow \Diamond p$	T(8) $\Box p \rightarrow Pp$
3 $\Box p \rightarrow p$	T(3) $p \rightarrow \Diamond p$	9 $Op \rightarrow \Box Op$	T(9) $\Diamond Pp \rightarrow Pp$
4 $\Box p \rightarrow \Box\Box p$	T(4) $\Diamond\Diamond p \rightarrow \Diamond p$	10 $Pp \rightarrow \Box Pp$	T(10) $\Diamond Op \rightarrow Op$
5 $p \rightarrow \Box\Diamond p$	T(5) $\Diamond\Box p \rightarrow p$	11 $O\Box p \rightarrow \Box Op$	T(11) $\Diamond Pp \rightarrow P\Diamond p$
6 $\Diamond p \rightarrow \Box\Diamond p$	T(6) $\Diamond\Box p \rightarrow \Box p$	12 $\Box Op \rightarrow O\Box p$	T(12) $P\Diamond p \rightarrow \Diamond Pp$

Table 4

Theorem 14. *Let* **S** *be a normal alethic-deontic system. Then* n *is a theorem in* **S** *if and only if* T(n) *is a theorem in* **S** (*for* $1 \leq n \leq 12$ *in table 4*).

Proof. This follows immediately from the duality corollary part (iii). In every case n has the form $MA \rightarrow NA$ and T(n) the form $D(N)A \rightarrow D(M)A$. ■

Comment 15. In the table 4 I have called T(n) “dual” theorems since T(n) may be derived from n (for $1 \leq n \leq 12$) by Dual. However, in a strict sense T(n) is of course not the dual of n.

4.1.4 More rules of inference in MADL

I will end this section by proving a set of new derived rules that are admissible in every ad system (theorem 16). First, we will introduce some new concepts.

All rules that can be derived in **dK** and in **aK** also hold in **MADL**. Some of these rules have a similar form as is easy to see. Let ⊗ be any of the following operators: O, P, □ or ◇. Then every rule of the following kind holds in **MADL**: if $\vdash A \rightarrow B$, then $\vdash \otimes A \rightarrow \otimes B$. Let us call a rule of this kind a *monotic rule of type I* (a MI rule). Let ⊗ be any of the following operators: F, U, ⬖ or ⊟. Then every rule of the following kind holds in **MADL**: if $\vdash A \rightarrow B$, then $\vdash \otimes B \rightarrow \otimes A$. We shall say that a rule of this kind is a *monotonic rule of type II* (or a converse monotonic rule) (a MII rule). Both type I and type II rules are called *monotonic*.

Theorem 16 (The inference rule theorem I). Let **S** be a normal ad system and let M and N be affirmative OP□◇ modalities. By an *affirmative* OP□◇ *modality* we mean a modality, i.e. a finite sequence, possibly empty, of the operators ¬, O, P, □ and ◇, in which ¬ occurs an even number of times (including zero). Then the following sentence is a theorem in **S**: A = MA → NA if and only if **S** has any of the following theorems or rules of inference: A’ = D(N) → D(M)A, (R1) if $\vdash_S A \rightarrow B$ then $\vdash_S MA \rightarrow NB$, or (R2) if $\vdash_S A \rightarrow B$, then $\vdash_S D(N)A \rightarrow D(M)B$.

Proof. We assume throughout that **S** is a normal ad system. To prove this theorem it is sufficient to establish how to obtain (i) A’ from A, (ii) A from A’, (iii) (R1) from A, (iv) A from (R1), (v) (R2) from (R1), and (vi) (R1) from (R2), in **S**. (ConP = Conditional Proof.)

Part (i) and **part (ii)** follow directly from the duality corollary.

Part (iii). From A to (R1). We assume that **S** includes MA → NA and then show (R1): if $\vdash_S A \rightarrow B$, then $\vdash_S MA \rightarrow NB$.

1.	$\vdash_S A \rightarrow B$	[Assumption]
2.	$\vdash_S NA \rightarrow NB$	[1, Repeated applications of MI rules]
3.	$\vdash_S MA \rightarrow NA$	[Given]
4.	$\vdash_S MA \rightarrow NB$	[2, 3, PL]
5.	If $\vdash_S A \rightarrow B$, then $\vdash_S MA \rightarrow NB$.	[ConP 1–4]

Part (iv). From (R1) to A. We assume that (R1), if $\vdash_S A \rightarrow B$ then $\vdash_S MA \rightarrow NB$, is a rule of inference in **S** and then prove A: $MA \rightarrow NA$.

1. $\vdash_S A \rightarrow A$ [PL]
2. $\vdash_S MA \rightarrow NA$ [1, R1]

Part (v). From (R1) to (R2). Suppose that (R1), if $\vdash_S A \rightarrow B$ then $\vdash_S MA \rightarrow NB$, is a rule of inference in **S**. We must prove that (R2), if $\vdash_S A \rightarrow B$ then $\vdash_S D(N)A \rightarrow D(M)B$, is a rule of inference in **S** too.

1. $\vdash_S A \rightarrow B$ [Assumption]
2. $\vdash_S A \rightarrow A$ [PL]
3. $\vdash_S MA \rightarrow NA$ [2, R1]
4. $\vdash_S D(N)A \rightarrow D(M)A$ [3, Dual corollary]
5. $\vdash_S D(M)A \rightarrow D(M)B$ [1, Repeated applications of MI rules]
6. $\vdash_S D(N)A \rightarrow D(M)B$ [4, 5, PL]
7. If $\vdash_S A \rightarrow B$, then $\vdash_S D(N)A \rightarrow D(M)B$ [1–6, ConP]

Part (vi). From (R2) to (R1). We suppose that **S** includes (R2), if $\vdash_S A \rightarrow B$ then $\vdash_S D(N)A \rightarrow D(M)B$, and then show that (R1), if $\vdash_S A \rightarrow B$ then $\vdash_S MA \rightarrow NB$, is included in **S** too.

1. $\vdash_S A \rightarrow B$ [Assumption]
2. $\vdash_S A \rightarrow A$ [PL]
3. $\vdash_S D(N)A \rightarrow D(M)A$ [2, R2]
4. $\vdash_S MA \rightarrow NA$ [3, Dual corollary]
5. $\vdash_S NA \rightarrow NB$ [1, Repeated applications of MI rules]
6. $\vdash_S MA \rightarrow NB$ [4, 5, PL]
7. If $\vdash_S A \rightarrow B$, then $\vdash_S MA \rightarrow NB$ [1–6, ConP] ■

Example 17. The following examples are consequences or instances of theorem 16. (i) If **S** includes $OA \rightarrow \Diamond A$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S OA \rightarrow \Diamond B$. (ii) If **S** includes $\Box A \rightarrow OA$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S \Box A \rightarrow OB$. (iii) If **S** includes $OA \rightarrow \Diamond A$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S \Box A \rightarrow PB$. (iv) If **S** includes $\Box A \rightarrow OA$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S PA \rightarrow \Diamond B$. (See sections 4.4.3 and 4.5.3.) (v) If **S** includes $OA \rightarrow \Box OA$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S OA \rightarrow \Box OB$. (vi) If **S** includes $PA \rightarrow \Box PA$, then: if $\vdash_S A \rightarrow B$, then $\vdash_S PA \rightarrow \Box PB$.

We will now begin to consider some extensions of **MADL**.

4.2 aKDdKad∅

aKDdKad∅, the smallest normal alethic-deontic logic that includes the axiom aD (i.e. the sentence $\Box p \rightarrow \Diamond p$), is the same system as **MADL** + {aD}. We will also call this system **S2**. Like every normal alethic-deontic system **aKDdKad∅** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. Every normal alethic-deontic system that includes aD is a normal **aKDdKad∅**-system. In other words, any normal alethic-deontic system that is an extension of **aKDdKad∅** is a normal **aKDdKad∅**-system. Since **aKDdKad∅** does not contain any mixed axioms, any axioms that contain both alethic and deontic operators, it is an ad combination. More precisely, it is an ad combination of the purely alethic system **aKD** and the smallest normal deontic system **OK**.

We will now consider what the ad octagon looks like in this system.

4.2.1 The alethic-deontic octagon

Figure 2 is a picture of the alethic-deontic octagon in **aKDdKad∅**. It is interpreted in the same way as the ad octagon in section 4.1.

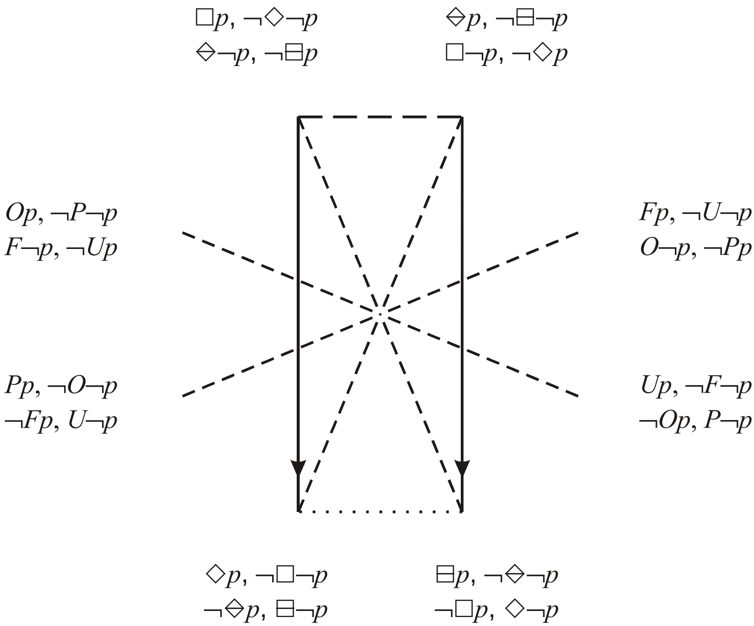

Figure 2. The Alethic-Deontic Octagon, MADL + {aD} **(S2).**

A dashed line connects sentences that are contradictories, e.g. Op and P¬p, and ⟠p and ◇p. An arrow from a sentence, A, to another sentence, B, indicates that A implies B; e.g. □p→¬□¬p and ⟠p→◇¬p are theorems in **MADL** + {aD}. A dotted line between two sentences, A and B, represents the fact that A and B are subcontraries, e.g. we can prove that ◇p∨◇¬p and ¬⟠p∨¬□p are theorems in the current system. Finally, a dotted line with long dots between two sentences, A and B, indicates that A and B are contraries, for instance □p and ⟠p, and ⟠¬p and ¬◇p; i.e. ¬(□p∧⟠p) and ¬(⟠¬p∧¬◇p) are theorems in **MADL** + {aD} (see Rönnedal (2010) for more on these concepts). We state this result formally.

Theorem 18. *All of the relationships displayed in figure 2 hold in every normal alethic-deontic* **aKDdKad∅***-system.*

Proof. This follows immediately from the fact that **aKDdKad∅** includes **OK** and **aKD**. ■

Remark 19. Note that a **aKDdKad∅**-system that is a proper extension of **aKDdKad∅** may contain more relations than those displayed in figure 2. The system **aKDdKDad∅** is, for instance, a proper extension of **aKDdKad∅**. The ad octagon for this system is displayed in figure 7. As can be seen, this system includes e.g. the sentences Op → Pp and Fp → P¬p, which are not theorems in **aKDdKad∅**. However, no **aKDdKad∅**-system lacks any of the theorems indicated in figure 2. Similar remarks apply to several other theorems involving the ad octagon stated in this paper.

4.3 aKdKDad∅

aKdKDad∅ is another example of an ad combination. It is identical to **aKdSDLad∅**, i.e. to the ad combination of the purely alethic system **aK** (see Chellas (1980)) and the purely deontic system Standard deontic logic (**SDL**) (see Rönnedal (2010)). In other words, **aKdKDad∅** is the smallest normal alethic-deontic logic that includes the axiom dD, i.e. the sentence Op→Pp. Accordingly, **aKdKDad∅** = **MADL** + {dD}. We will also call this system **S3**. Since it is a normal alethic-deontic system **aKdKDad∅** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. A normal **aKdKDad∅**-system is any normal alethic-deontic system that includes dD, or in other words, any normal alethic-deontic system that is an extension of **aKdKDad∅**.

Let us consider some properties of this system.

4.3.1 The alethic-deontic octagon

The alethic-deontic octagon in **aKdKDad∅** is similar to the ad octagon in **aKDdKad∅**. The differences are due to the fact that **aKdKDad∅** includes dD while **aKDdKad∅** includes aD. The similarities depend upon the formal similarities between these axioms.

Theorem 20. *All of the relationships displayed in figure 3 hold in every normal alethic-deontic* **aKdKDad∅***-system.*

Proof. This follows immediately from the fact that **aKdKDad∅** includes **SDL** and **aK**. ■

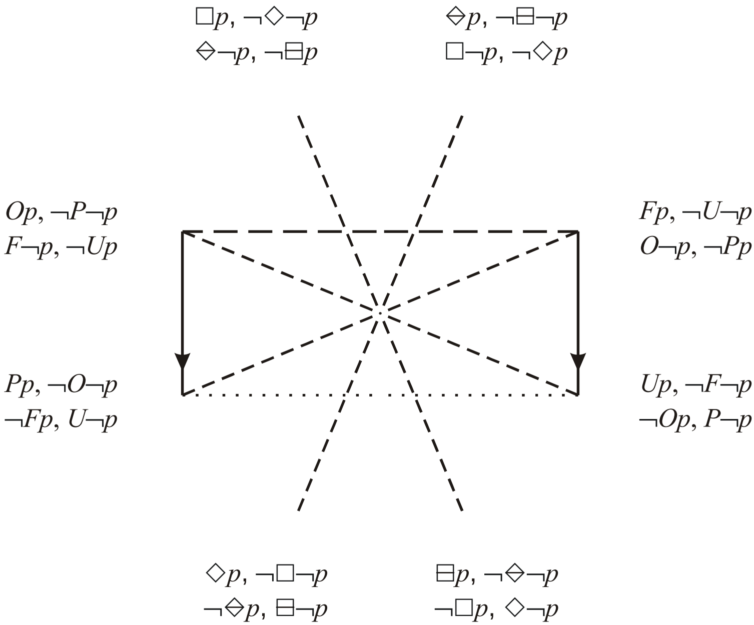

Figure 3. The Alethic-Deontic Octagon, MADL + {dD} **(S3).**

Next we turn to two ad systems that include mixed axioms: **aKdKadOC** and **aKdKadMO**. OC and MO are two of the most interesting mixed axioms we will consider.

4.4 aKdKadOC

aKdKadOC is the smallest normal alethic-deontic logic that contains the axiom OC, i.e. the sentence Op → ◇p. Therefore, **aKdKadOC** = **MADL** + {OC}. We will also call this system **S7**. Since it is a normal alethic-deontic system **aKdKadOC** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. A normal **aKdKadOC**-system is any normal alethic-deontic system that includes OC, or in other words, any normal alethic-deontic system that is an extension of **aKdKadOC**.

Let us consider some properties of this system.

First of all we note that O(Op → ◇p) is a theorem in **aKdKadOC**. OC′ follows immediately from OC by O-necessitation. Accordingly, OC′ is a theorem in every **aKdKadOC**-system.

Next we turn to the alethic-deontic octagon in **aKdKadOC**.

4.4.1 The alethic-deontic octagon in aKdKadOC

Every system considered so far has been an ad combination, i.e. a combination of a purely alethic and a purely deontic system (see above). However, **aKdKadOC** is not a system of this kind, since OC includes both alethic and deontic operators. This is an example of a mixed axiom. When OC is added to **MADL** several interesting theorems follow. Figure 4 displays the relationships between primary alethic and deontic sentences in **aKdKadOC**.

Theorem 21. *All of the relationships displayed in figure 4 hold in every normal alethic-deontic* **aKdKadOC**-*system.*

Proof. Most of the proofs are quite easy and are left to the reader. (Table 5 includes a list of some of the theorems that are displayed in figure 4.) ■

	Theorems	
□p → Pp	¬(□p ∧ Fp)	Pp ∨ ◇¬p
Fp → ◇¬p	¬(Op ∧ ⟠p)	P¬p ∨ ◇p
Fp → ¬□p	¬(Op ∧ □¬p)	Pp ∨ ¬□p
⟠p → ¬Op	¬(O¬p ∧ □p)	¬Op ∨ ◇p
⟠p → P¬p	¬(O¬p ∧ ⟠¬p)	¬O¬p ∨ ◇¬p
Fp → ⊟p	¬(F¬p ∧ □¬p)	¬Fp ∨ ◇¬p
⟠p → Up	¬(Fp ∧ ⟠¬p)	¬F¬p ∨ ◇p

Table 5

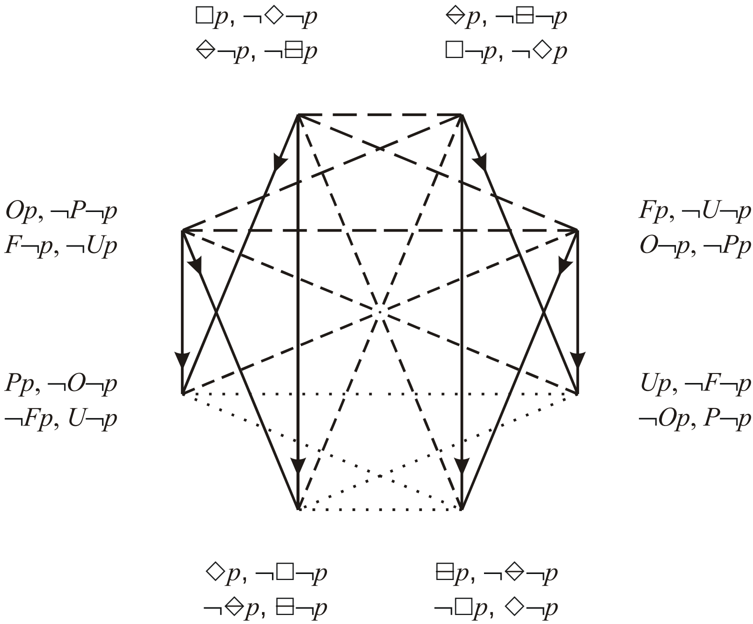

Figure 4. The Alethic-Deontic Octagon, MADL + {OC} **(S7).**

Note that all of the sentences in table 5 are equivalent in the system **aKdKadOC**. So, **MADL** + any sentence in table 5 is deductively equivalent with **aKdKadOC**. All sentences in table 5 stand or fall together. If we accept one of them we should accept all the others and if we reject one, we should reject all the others.

Since OC is one interpretation of the so-called ought-can principle, **aKdKadOC** tells us something about what follows by accepting this principle.

4.4.2 Some theorems including necessary implications

I will soon establish some derived rules in **aKdKadOC**. But first I will consider some theorems that include necessary implications.

	Theorem	Intuitive reading
(i)	$\Box(p \to q) \to (Op \to \Diamond q)$	If it is necessary that p implies q, then it is possible that q if it is obligatory that p.

(ii)	$(Op \wedge \Box(p \rightarrow q)) \rightarrow \Diamond q$	If it is obligatory that p and it is necessary that p implies q, then it is possible that q.
(iii)	$Op \rightarrow (\Box(p \rightarrow q) \rightarrow \Diamond q)$	If it is obligatory that p, then it is possible that q if it is necessary that p implies q.
(iv)	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow Pq)$	If it is necessary that p implies q, then it is permitted that q if it is necessary that p.
(v)	$(\Box p \wedge \Box(p \rightarrow q)) \rightarrow Pq$	If it is necessary that p and it is necessary that p implies q, then it is permitted that q.
(vi)	$\Box p \rightarrow (\Box(p \rightarrow q) \rightarrow Pq)$	If it is necessary that p, then it is permitted that q if it is necessary that p implies q.
(vii)	$\Box(p \rightarrow q) \rightarrow (Fq \rightarrow \boxminus p)$	If it is necessary that p implies q, then it is unnecessary that p if it is forbidden that q.
(viii)	$(Fq \wedge \Box(p \rightarrow q)) \rightarrow \boxminus p$	If it is forbidden that q and it is necessary that p implies q, then it is unnecessary that p.
(ix)	$Fq \rightarrow (\Box(p \rightarrow q) \rightarrow \boxminus p)$	If it is forbidden that q, then it is unecessary that p if it is necessary that p implies q.
(x)	$\Box(p \rightarrow q) \rightarrow (\diamondminus q \rightarrow Up)$	If it is necessary that p implies q, then it is unobligatory that p if it is impossible that q.
(xi)	$(\diamondminus q \wedge \Box(p \rightarrow q)) \rightarrow Up$	If it is impossible that q and it is necessary that p implies q, then it is unobligatory that p.
(xii)	$\diamondminus q \rightarrow (\Box(p \rightarrow q) \rightarrow Up)$	If it is impossible that q, then it is unobligatory that p if it is necessary that p implies q.

Table 6

Theorem 22. *All sentences in table 6 are theorems in* **aKdKadOC**.

Proof. Part (ii) and part (iii) follow immediately from part (i) by PL. Likewise part (v) and part (vi) follow from part (iv), part (viii) and part (ix) from part (vii), and part (xi) and part (xii) from part (x), all by PL. So, we only have to show part (i), part (iv), part (vii) and part (x). I will leave part (vii) and part (x) to the reader and prove the rest.

Part (i). $\Box(p \rightarrow q) \rightarrow (Op \rightarrow \Diamond q)$

1.	$\Box(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$	[**aK**]
2.	$Op \rightarrow \Diamond p$	[OC]
3.	$\Box(p \rightarrow q) \rightarrow (Op \rightarrow \Diamond q)$	[1, 2, PL]

Step (1) is a theorem in the minimal alethic modal system **aK**. So, it is a theorem in every normal alethic and alethic-deontic system. We have indicated this by writing **aK** in the justificatory entry. Part (i) says that if it is necessary that p implies q, then if it is obligatory that p then it is possible that q.

Part (iv). $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow Pq)$

1.	□(p → q) → (□p →□q)	[aK]
2.	□q → Pq	[T21 q/p]
3.	□(p → q) → (□p →Pq)	[1, 2, PL]

Part (iv) claims that if it is necessary that p implies q, then if it is necessary that p then it is permitted that q. Step (1) is the axiom aK, step (2) is obtained from theorem 21 by substituting q for p and step (3) is deduced from (1) and (2) by PL. ■

We are now in a position to prove some derived rules.

4.4.3 Some derived rules in aKdKadOC

	Derived Rules	**Intuitive reading**
(i)	If ⊢ A → B, then ⊢ OA → ◇B	If A implies B is a theorem, then OA implies ◇B is a theorem.
(ii)	If ⊢ A → B, then ⊢ □A → PB	If A implies B is a theorem, then □A implies PB is a theorem.
(iii)	If ⊢ A → B, then ⊢ FB → ⊟A	If A implies B is a theorem, then FB implies ⊟B is a theorem.
(iv)	If ⊢ A → B, then ⊢ ⟠B → UA	If A implies B is a theorem, then ⟠B implies UA is a theorem.

Table 7

Theorem 23. *All rules in table 7 are derived rules in* **aKdKadOC**.

Proof. I will prove part (i) and part (ii), part (iii) and part (iv) are left to the reader.

Derived rule (i). If ⊢ A → B, then ⊢ OA → ◇B. If A implies B is a theorem, then OA implies ◇B is a theorem.

Proof. Suppose (1) ⊢ A → B. Then, (2) ⊢ □(A → B) [from 1 by □-necessitation]. Hence, (3) ⊢ OA → ◇B [from 2 and theorem 22]. It follows that (4) if ⊢ A → B, then ⊢ OA → ◇B [by conditional proof from 1–3 discharging the assumption].

Derived rule (ii). If ⊢ A → B, then ⊢ □A → PB. If A implies B is a theorem, then □A implies PB.

Proof. Suppose (1) ⊢ A → B. Then (2) ⊢ □(A → B) [from 1 by □-necessitation]. (3) ⊢ □(A → B) → (□A → PB) [by theorem 22]. So, (4) ⊢ □A → PB [from 2 and 3 by MP]. Consequently, (5) if ⊢ A → B, then ⊢ □A → PB [by conditional proof from 1–4 discharging the assumption]. ■

4.4.4 Conjunctive and disjunctive obligations, permissions, necessities and possibilities

Let us consider some theorems that include conjunctive and disjunctive obligations, permissions, necessities and possibilities.

	Theorem	Informal reading
(i)	$(Op \wedge Oq) \rightarrow \Diamond(p \wedge q)$	If it is obligatory that p and it is obligatory that q, then it is possible that p and q.
(ii)	$(\Box p \wedge \Box q) \rightarrow P(p \wedge q)$	If both p and q are necessary, then it is permitted that p and q.
(iii)	$(Op \vee Oq) \rightarrow \Diamond(p \vee q)$	If it is obligatory that p or it is obligatory that q, then it is possible that p or q.
(iv)	$(\Box p \vee \Box q) \rightarrow P(p \vee q)$	If either p or q is necessary, then it is permitted that p or q.
(v)	$O(p \wedge q) \rightarrow (\Diamond p \wedge \Diamond q)$	If it is obligatory that p and q, then both p and q are possible.
(vi)	$\Box(p \wedge q) \rightarrow (Pp \wedge Pq)$	If it is necessary that p and q, then it is permitted that p and it is permitted that q.

Table 8

Theorem 24. *Every sentences in table 8 is a theorem in* **aKdKadOC**.

Proof. Straightforward. Rönnedal (2010) may be helpful. ■

In section 4.4.6 we will see how to generalise this theorem.

4.4.5 More rules

Let us consider some more derived rules.

	Derived Rule	
(i)	If $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash (OA_1 \vee ... \vee OA_n) \rightarrow \Diamond A$	(for $n \geq 0$)
(ii)	If $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash \diamondminus A \rightarrow (UA_1 \wedge ... \wedge UA_n)$	(for $n \geq 0$)
(iii)	If $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash (\Box A_1 \vee ... \vee \Box A_n) \rightarrow PA$	(for $n \geq 0$)
(iv)	If $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash FA \rightarrow (\boxminus A_1 \wedge ... \wedge \boxminus A_n)$	(for $n \geq 0$)
(v)	If $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash (OA_1 \wedge ... \wedge OA_n) \rightarrow \Diamond A$	(for $n \geq 0$)
(vi)	If $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash \diamondminus A \rightarrow (UA_1 \vee ... \vee UA_n)$	(for $n \geq 0$)
(vii)	If $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash (\Box A_1 \wedge ... \wedge \Box A_n) \rightarrow PA$	(for $n \geq 0$)
(viii)	If $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash FA \rightarrow (\boxminus A_1 \vee ... \vee \boxminus A_n)$	(for $n \geq 0$)
(ix)	If $\vdash A \rightarrow (A_1 \vee ... \vee A_n)$, then $\vdash OA \rightarrow (\Diamond A_1 \vee ... \vee \Diamond A_n)$	(for $n \geq 0$)
(x)	If $\vdash A \rightarrow (A_1 \vee ... \vee A_n)$, then $\vdash (\diamondminus A_1 \wedge ... \wedge \diamondminus A_n) \rightarrow UA$	(for $n \geq 0$)
(xi)	If $\vdash A \rightarrow (A_1 \vee ... \vee A_n)$, then $\vdash \Box A \rightarrow (PA_1 \vee ... \vee PA_n)$	(for $n \geq 0$)

(xii)	If $\vdash A \rightarrow (A_1 \vee ... \vee A_n)$, then $\vdash (FA_1 \wedge ... \wedge FA_n) \rightarrow \boxminus A$	(for $n \geq 0$)
(xiii)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash OA \rightarrow (\Diamond A_1 \wedge ... \wedge \Diamond A_n)$	(for $n \geq 0$)
(xiv)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash (\diamondminus A_1 \vee ... \vee \diamondminus A_n) \rightarrow UA$	(for $n \geq 0$)
(xv)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash \Box A \rightarrow (PA_1 \wedge ... \wedge PA_n)$	(for $n \geq 0$)
(xvi)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash (FA_1 \vee ... \vee FA_n) \rightarrow \boxminus A$	(for $n \geq 0$)

Table 9

Theorem 25. *All rules in table 9 are derived rules in* **aKdKadOC**.

Proof. Left to the reader. Rönnedal (2010) may be helpful. ■

4.4.6 Generalisations of distribution theorems

	Theorem		Theorem
(i)	$(Op_1 \wedge ... \wedge Op_n) \rightarrow \Diamond(p_1 \wedge ... \wedge p_n)$	(iv)	$(\Box p_1 \vee ... \vee \Box p_n) \rightarrow P(p_1 \vee ... \vee p_n)$
(ii)	$(\Box p_1 \wedge ... \wedge \Box p_n) \rightarrow P(p_1 \wedge ... \wedge p_n)$	(v)	$O(p_1 \wedge ... \wedge p_n) \rightarrow (\Diamond p_1 \wedge ... \wedge \Diamond p_n)$
(iii)	$(Op_1 \vee ... \vee Op_n) \rightarrow \Diamond(p_1 \vee ... \vee p_n)$	(vi)	$\Box(p_1 \wedge ... \wedge p_n) \rightarrow (Pp_1 \wedge ... \wedge Pp_n)$

Table 10

Theorem 26. *All sentences of the forms in table 10 are theorems in* **aKdKadOC**.

Proof. **Part (i)**. $(Op_1 \wedge ... \wedge Op_n) \rightarrow \Diamond(p_1 \wedge ... \wedge p_n)$.

1. $(p_1 \wedge ... \wedge p_n) \rightarrow (p_1 \wedge ... \wedge p_n)$ [PL]
2. $(Op_1 \wedge ... \wedge Op_n) \rightarrow \Diamond(p_1 \wedge ... \wedge p_n)$ [1, T25(v)]

Part (i) says that if it is obligatory that p_1 and … and it is obligatory that p_n, then it is possible that p_1 and … and p_n. So, a conjunction is possible if each conjunct is obligatory. The proof uses T25(v): if $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash (OA_1 \wedge ... \wedge OA_n) \rightarrow \Diamond A$. Note that the sentence on line (1) is of the form $(A_1 \wedge ... \wedge A_n) \rightarrow A$. An alternative proof uses O-distribution and OC like this.

1. $O(p_1 \wedge ... \wedge p_n) \rightarrow \Diamond(p_1 \wedge ... \wedge p_n)$ [OC, $p_1 \wedge ... \wedge p_n/p$]
2. $(Op_1 \wedge ... \wedge Op_n) \rightarrow \Diamond(p_1 \wedge ... \wedge p_n)$ [1, Dist]

See Rönnedal (2010) for more on how various deontic operators distribute.

Part (ii). $(\Box p_1 \wedge ... \wedge \Box p_n) \rightarrow P(p_1 \wedge ... \wedge p_n)$.

Part (ii) claims that a conjunction is permitted if each conjunct is necessary. In other words, if it is necessary that p_1 and … and it is necessary that p_n, then it is permitted that p_1 and … and p_n. The proof is similar to the proof of part (i), but this time use T25(vii): if $\vdash (A_1 \wedge ... \wedge A_n) \rightarrow A$, then $\vdash (\Box A_1 \wedge ... \wedge \Box A_n) \rightarrow PA$.

Part (iii). $(Op_1 \vee ... \vee Op_n) \rightarrow \Diamond(p_1 \vee ... \vee p_n)$.

According to part (iii) a disjunction is possible if any disjunct is obligatory. That is, if it is obligatory that p_1 or … or it is obligatory that p_n, then it is possible that p_1 or … or p_n. The proof is similar to the proof of part (i) but uses T25(i): if $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash (OA_1 \vee ... \vee OA_n) \rightarrow \Diamond A$.

Part (iv). $(\Box p_1 \vee ... \vee \Box p_n) \rightarrow P(p_1 \vee ... \vee p_n)$.

Part (iv) asserts that if it is necessary that p_1 or … or it is necessary that p_n, then it is permitted that p_1 or … or p_n. So, a disjunction is permitted if any disjunct is necessary. The sentence follows directly from PL and T25(iii): if $\vdash (A_1 \vee ... \vee A_n) \rightarrow A$, then $\vdash (\Box A_1 \vee ... \vee \Box A_n) \rightarrow PA$.

Part (v). $O(p_1 \wedge ... \wedge p_n) \rightarrow (\Diamond p_1 \wedge ... \wedge \Diamond p_n)$.

Part (v) follows immediately from PL and T25(xiii): if $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash OA \rightarrow (\Diamond A_1 \wedge ... \wedge \Diamond A_n)$. According to the sentence each conjunct is possible if a conjunction is obligatory. I.e. if it is obligatory that p_1 and … and p_n, then it is possible that p_1 and … and it is possible that p_n.

Part (vi). $\Box(p_1 \wedge ... \wedge p_n) \rightarrow (Pp_1 \wedge ... \wedge Pp_n)$.

Part (vi) says that it is permitted that p_1 and … and it is permitted that p_n if it is necessary that p_1 and … and p_n. So, if a conjunction is necessary, then each conjunct is permitted. The proof is similar to the proof of part (i) but uses T25(xv): if $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash \Box A \rightarrow (PA_1 \wedge ... \wedge PA_n)$. ■

Theorem 27. (i) **MADL** + OC *includes* OC′, $\Box p \rightarrow \Diamond p$ *and* $Op \rightarrow Pp$. (ii) *All of the following systems are deductively equivalent:* **aKdKadOC**, **aKDdKadOC**, **aKdKDadOC** *and* **aKDdKDadOC**.

Proof. Left to the reader. ■

4.5 aKdKadMO

The smallest normal alethic-deontic logic that includes the axiom MO, i.e. the sentence $\Box p \rightarrow Op$, is **aKdKadMO**. Accordingly, **aKdKadMO** = **MADL** + {MO}. We will also call this system **S4**. Since it is a normal alethic-deontic system **aKdKadMO** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. A normal **aKdKadMO**-system is any normal alethic-deontic system that includes MO, or in other words, any normal alethic-deontic system that is an extension of **aKdKadMO**.

Let us consider some properties of this system.

4.5.1 The alethic-deontic octagon

Figure 5 displays the alethic-deontic octagon in the system **aKdKadMO**. The octagon is interpreted as usual.

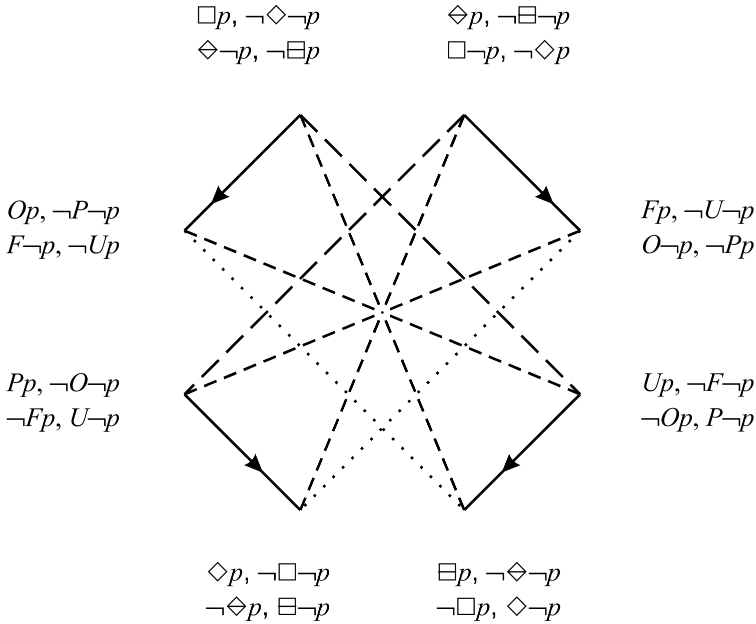

Figure 5. The Alethic-Deontic Octagon, MADL + {MO} **(S4).**

Theorem 28. *All of the relationships displayed in figure 5 hold in every normal alethic-deontic* **aKdKadMO***-system.*

Proof. Most of the proofs are quite easy and are left to the reader. Table 11 includes a list of some theorems displayed in figure 5. Note that all sentences in this table are equivalent in **aKdKadMO**. **MADL** + any sentence in table 11 is deductively equivalent with **MADL** + {MO}. ■

	Theorems	
Pp → ◇p	¬(□p ∧ ¬Op)	Fp ∨ ◇p
⟠p → Fp	¬(Pp ∧ □¬p)	Op ∨ ◇¬p
¬Op → ¬□p	¬(□p ∧ P¬p)	O¬p ∨ ◇p
⟠p → ¬Pp	¬(Pp ∧ ⟠p)	Op ∨ ¬□p
□p → F¬p	¬(⟠¬p ∧ ¬Op)	F¬p ∨ ◇¬p
Pp→¬⟠p	¬(¬Fp ∧ ⟠p)	Fp ∨ ¬□¬p
Up → ⊟p	¬(□p ∧ ¬F¬p)	¬Pp ∨ ◇p

Table 11

4.5.2 The means-end principle

Perhaps the most interesting feature of the system **aKdKadMO** is that the so-called means-end principle is a theorem in it. According to the means-end principle every necessary consequence of what ought to be ought to be. Table 12 includes this principle and several similar theorems.

	Theorem	**Informal reading**
(i)	$\Box(p \rightarrow q) \rightarrow (Op \rightarrow Oq)$	If it is necessary that p implies q, then if it ought to be that p then it ought to be that q.
(ii)	$(Op \land \Box(p \rightarrow q)) \rightarrow Oq$	If it ought to be that p and it is necessary that if p then q, then it ought to be that q.
(iii)	$Op \rightarrow (\Box(p \rightarrow q) \rightarrow Oq)$	If it ought to be that p, then if it is necessary that p implies q then it ought to be that q.
(iv)	$\Box(p \rightarrow q) \rightarrow (Pp \rightarrow Pq)$	If it is necessary that p implies q, then if it is permitted that p it is permitted that q.
(v)	$(Pp \land \Box(p \rightarrow q)) \rightarrow Pq$	If it is permitted that p and it is necessary that p implies q, then q is permitted.
(vi)	$Pp \rightarrow (\Box(p \rightarrow q) \rightarrow Pq)$	If it is permitted that p, then if it is necessary that p implies q then q is permitted.
(vii)	$\Box(p \rightarrow q) \rightarrow (Fq \rightarrow Fp)$	If it is necessary that p implies q, then if it is forbidden that q then it is forbidden that p.
(viii)	$(Fq \land \Box(p \rightarrow q)) \rightarrow Fp$	If it is forbidden that q and it is necessary that p implies q, then it is forbidden that p.
(ix)	$Fq \rightarrow (\Box(p \rightarrow q) \rightarrow Fp)$	If it is forbidden that q, then if it is necessary that p implies q then it is forbidden that p.
(x)	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow Oq)$	If it is necessary that p implies q, then it is obligatory that q if it is necessary that p.
(xi)	$(\Box p \land \Box(p \rightarrow q)) \rightarrow Oq$	If it is necessary that p and it is necessary that p implies q, then it is obligatory that q.
(xii)	$\Box p \rightarrow (\Box(p \rightarrow q) \rightarrow Oq)$	If it is necessary that p, then if it is necessary that p implies q it is obligatory that q.
(xiii)	$\Box(p \rightarrow q) \rightarrow (Pp \rightarrow \Diamond q)$	If it is necessary that p implies q, then it is possible that q if it is permitted that p.
(xiv)	$(Pp \land \Box(p \rightarrow q)) \rightarrow \Diamond q$	If it is permitted that p and it is necessary that p implies q, then it is possible that q.
(xv)	$Pp \rightarrow (\Box(p \rightarrow q) \rightarrow \Diamond q)$	If it is permitted that p, then if it is necessary that p implies q it is possible that q.
(xvi)	$\Box(p \rightarrow q) \rightarrow (⟐q \rightarrow Fp)$	If it is necessary that p implies q, then it is forbidden that p if it is impossible that q.

(xvii)	(⟠q ∧ □(p → q)) → Fp	If it is impossible that q and it is necessary that p implies q, then it is forbidden that p.
(xviii)	⟠q → (□(p → q) → Fp)	If it is impossible that q, then if it is necessary that p implies q it is forbiden that p.
(xix)	□(p → q) → (Uq → ⊟p)	If it is necessary that p implies q, then it is unnecessary that p if it is unobligatory that q.
(xx)	(Uq ∧ □(p → q)) → ⊟p	If it is unobligatory that q and it is necessary that p implies q, then it is unnecessary that p.
(xxi)	Uq → (□(p → q) → ⊟p)	If it is unobligatory that q, then if it is necessary that p implies q it is unnecessary that p.

Table 12

Theorem 29. *All sentences in table 12 are theorems in* **aKdKadMO**.

Proof. I will prove part (i), part (vii), part (xiii) and part (xvi) to illustrate the method, and leave the rest to the reader. The philosophically most interesting parts are perhaps part (i) – (ix). These theorems can be used to derive obligations from obligations, permissions from permissions and prohibitions from prohibitions, with the help of necessary implications.

Part (i). □(p→q)→(Op→Oq)

1.	O(p→q)→(Op→Oq)	[dK]
2.	□(p→q)→O(p→q)	[MO, p→q/p]
3.	□(p→q)→(Op→Oq)	[1, 2, PL]

Part (i) is one version of the means-end principle. Part (ii) and (iii) are similar versions of this principle. It is easy to derive part (ii) and part (iii) from part (i). The means-end principle is intuitively plausible and can be very useful when deriving "new" obligations from "old" obligations. Suppose for instance that it ought to be that everyone is honest. Then it follows that you ought to be honest. For it is necessary that if everyone is honest then you are honest.

Part (vii). □(p→q)→(Fq→Fp)

1.	O(p→q)→(Fq→Fp)	[**OK**]
2.	□(p→q)→O(p→q)	[MO, p→q/p]
3.	□(p→q)→(Fq→Fp)	[1, 2, PL]

Note that the sentence at step (1) is provable in the deontic system **OK**; and since every normal ad system includes **OK**, this sentence is a theorem in **MADL** + MO too. Part (vii) is also a quite useful principle. Suppose it is forbidden that you smoke in this restaurant. Then it follows that it is forbidden that you smoke a cigar in this restaurant. For it is necessary that if you smoke a cigar in this restaurant you smoke in this restaurant.

Part (xiii). □(p→q)→(Pp→◇q)

1.	□(p→q)→O(p→q)	[MO, p→q/p]
2.	O(p→q)→(Pp→Pq)	[**OK**]
3.	Pq→◇q	[T28, q/p]
4.	□(p→q)→(Pp→◇q)	[1, 2, 3, PL]

Note that the sentence at step (2) is provable in the deontic system **OK**; and since every normal ad system includes **OK**, this sentence is a theorem in **MADL** + MO too.

Part (xvi). □(p→q)→(⟠q→Fp)

1.	□(p→q)→(Pp→◇q)	[(xiii)]
2.	(Pp→◇q)↔(⟠q→Fp)	[PL, adIT etc.]
3.	□(p→q)→(⟠q→Fp)	[1, 2, PL] ■

4.5.3 Some derived rules in aKdKadMO

	Derived Rules	Informal reading
(i)	If ⊢ A → B, then ⊢ □A → OB	If A implies B is a theorem, then □A implies OB is a theorem.
(ii)	If ⊢ A → B, then ⊢ PA → ◇B	If A implies B is a theorem, then PA implies ◇B is a theorem.
(iii)	If ⊢ A → B, then ⊢ ⟠B → FA	If A implies B is a theorem, then ⟠B implies FA is a theorem.
(iv)	If ⊢ A → B, then ⊢ UB → ⊟A	If A implies B is a theorem, then UB implies ⊟A is a theorem.

Table 13

Theorem 30. *All rules in table 13 are derived rules in* **aKdKadMO**.

Proof.

Derived rule (i). If ⊢ A → B, then ⊢ □A → OB. If A implies B is a theorem, then □A implies OB is a theorem.

Proof. Suppose that (1) ⊢ A → B. Then (2) ⊢ □(A → B) [by □-necessitation from 1]. (3) ⊢ □(A → B) → (□A → OB) [by theorem 29(x)]. Hence, (4) ⊢ □A → OB [from 2 and 3 by modus ponens]. It follows that (5) if ⊢ A → B, then ⊢ □A → OB [by conditional proof from 1–4 discharging the assumption].

Derived rule (ii). If ⊢ A → B, then ⊢ PA → ◇B. If A implies B is a theorem, then PA implies ◇B is a theorem.

Proof. Suppose (1) ⊢A → B. Then (2) ⊢ □(A → B) [from 1 by □-necessitation]. Hence, (3) ⊢ PA → ◇B [from 2 and theorem 29(xiii)]. It follows that (4) if ⊢ A → B, then ⊢ PA → ◇B [by conditional proof from 1–3 discharging the assumption].

Derived rule (iii). If ⊢ A → B, then ⊢ ⟠B → FA. If A implies B is a theorem, then ⟠B implies FA is a theorem.

Proof. Suppose that (1) ⊢A → B. Then (2) ⊢ □(A → B) [from 1 by □-necessitation]. Hence, (3) ⊢ ⟠B → FA [from 2 and theorem 29(xvi)]. It follows that (4) if ⊢ A → B, then ⊢ ⟠B → FA [by conditional proof from 1–3 discharging the assumption].

Derived rule (iv). If ⊢ A → B, then ⊢ UB → ⊟A. If A implies B is a theorem, then UB implies ⊟A is a theorem. Proof is left to the reader. ■

4.5.4 Conjunctive and disjunctive obligations, permissions, necessities and possibilities

Let us consider some theorems that include conjunctive and disjunctive obligations, permissions, necessities and possibilities.

	Theorem	Intuitive reading
(i)	(□p ∧ □q) → O(p ∧ q)	If both p and q are necessary, then it is obligatory that p and q.
(ii)	□(p ∧ q) → (Op ∧ Oq)	If it is necessary that p and q, then it is obligatory that p and it is obligatory that q.
(iii)	P(p ∧ q) → (◇p ∧ ◇q)	If it is permitted that p and q, then it is possible that p and it is possible that q.
(iv)	(⟠p ∨ ⟠q) → F(p ∧ q)	If it is impossible that p or it is impossible that q, then it is fobidden that p and q.
(v)	⟠(p ∨ q) → (Fp ∧ Fq)	If it is impossible that p or q, then both p and q are forbidden.
(vi)	(□p ∨ □q) → O(p ∨ q)	If it is necessary that p or it is necessary that q, then it is obligatory that p or q.
(vii)	(Pp ∨ Pq) → ◇(p ∨ q)	If it is permitted that p or it is permitted that q, then it is possible that p or q.
(viii)	P(p ∨ q) → (◇p ∨ ◇q)	If it is permitted that p or q, then either p or q is possible.
(ix)	(⟠p ∧ ⟠q) → F(p ∨ q)	If it is impossible that p and it is impossible that q, then it is forbidden that p or q.

Table 14

Theorem 31. *Every sentence in table 14 is a theorem in* **aKdKadMO**.

Proof. Straightforward. ■

4.5.5 The contingency octagon in aKdKadMO

It is possible to construct an alethic-deontic contingency octagon that displays the relationships between the alethic and deontic "contingency" operators. Figure 6 shows us how these concepts are related in **aKdKadMO**.

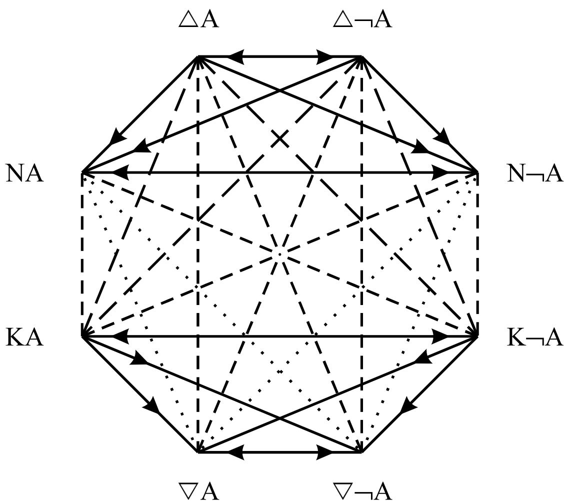

Figure 6. The Contingency Octagon in aKdKadMO.

4.5.6 More rules in aKdKadMO

	Derived Rules	
(i)	If $\vdash (A_1 \vee ... \vee A_n) \to A$, then $\vdash (\Box A_1 \vee ... \vee \Box A_n) \to OA$	(for $n \geq 0$)
(ii)	If $\vdash (A_1 \vee ... \vee A_n) \to A$, then $\vdash (PA_1 \vee ... \vee PA_n) \to \Diamond A$	(for $n \geq 0$)
(iii)	If $\vdash (A_1 \vee ... \vee A_n) \to A$, then $\vdash \diamondminus A \to (FA_1 \wedge ... \wedge FA_n)$	(for $n \geq 0$)
(iv)	If $\vdash (A_1 \vee ... \vee A_n) \to A$, then $\vdash UA \to (\boxminus A_1 \wedge ... \wedge \boxminus A_n)$	(for $n \geq 0$)
(v)	If $\vdash (A_1 \wedge ... \wedge A_n) \to A$, then $\vdash (\Box A_1 \wedge ... \wedge \Box A_n) \to OA$	(for $n \geq 0$)
(vi)	If $\vdash (A_1 \wedge ... \wedge A_n) \to A$, then $\vdash UA \to (\boxminus A_1 \vee ... \vee \boxminus A_n)$	(for $n \geq 0$)
(vii)	If $\vdash A \to (A_1 \vee ... \vee A_n)$, then $\vdash PA \to (\Diamond A_1 \vee ... \vee \Diamond A_n)$	(for $n \geq 0$)
(viii)	If $\vdash A \to (A_1 \vee ... \vee A_n)$, then $\vdash (\diamondminus A_1 \wedge ... \wedge \diamondminus A_n) \to FA$	(for $n \geq 0$)

(ix)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash \Box A \rightarrow (OA_1 \wedge ... \wedge OA_n)$	(for $n \geq 0$)
(x)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash PA \rightarrow (\Diamond A_1 \wedge ... \wedge \Diamond A_n)$	(for $n \geq 0$)
(xi)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash (⟠A_1 \vee ... \vee ⟠A_n) \rightarrow FA$	(for $n \geq 0$)
(xii)	If $\vdash A \rightarrow (A_1 \wedge ... \wedge A_n)$, then $\vdash (UA_1 \vee ... \vee UA_n) \rightarrow ⊟A$	(for $n \geq 0$)

Table 15

Theorem 32. *All rules in table 15 are derived rules in* **aKdKadMO**.

Proof. Left to the reader. ■

	Theorem		Theorem
(i)	$(\Box p_1 \wedge ... \wedge \Box p_n) \rightarrow O(p_1 \wedge ... \wedge p_n)$	(v)	$⟠(p_1 \vee ... \vee p_n) \rightarrow (Fp_1 \wedge ... \wedge Fp_n)$
(ii)	$\Box(p_1 \wedge ... \wedge p_n) \rightarrow (Op_1 \wedge ... \wedge Op_n)$	(vi)	$(\Box p_1 \vee ... \vee \Box p_n) \rightarrow O(p_1 \vee ... \vee p_n)$
(iii)	$P(p_1 \wedge ... \wedge p_n) \rightarrow (\Diamond p_1 \wedge ... \wedge \Diamond p_n)$	(vii)	$(Pp_1 \vee ... \vee Pp_n) \rightarrow \Diamond(p_1 \vee ... \vee p_n)$
(iv)	$(⟠p_1 \vee ... \vee ⟠p_n) \rightarrow F(p_1 \wedge ... \wedge p_n)$	(viii)	$P(p_1 \vee ... \vee p_n) \rightarrow (\Diamond p_1 \vee ... \vee \Diamond p_n)$
		(ix)	$(⟠p_1 \wedge ... \wedge ⟠p_n) \rightarrow F(p_1 \vee ... \vee p_n)$

Table 16

Theorem 33. *Every sentence in table 16 is a theorem in* **aKdKadMO**.

Proof. Straightforward. ■

4.6 aKDdKDad∅

The smallest normal alethic-deontic logic that includes the axioms aD and dD, i.e. the sentences $Op \rightarrow Pp$ and $\Box p \rightarrow \Diamond p$, is **aKDdKDad∅**. Consequently, **aKDdKDad∅** = **MADL** + {aD, dD}. It is our first example of an ad system that contains more than one additional axiom. Nevertheless, the system is an ad combination of the pure alethic logic **aKD** and the pure deontic logic **dKD** (**SDL**), since it doesn't contain any mixed axioms, in contrast to our two previous systems **aKdKadOC** and **aKdKadMO**. We will also call this system **S5**. **aKDdKDad∅** includes PL, the axioms aK and dK, the ordinary definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation, like every normal alethic-deontic system. We shall say that any normal alethic-deontic system that is an extension of **aKDdKDad∅**, i.e. any normal alethic-deontic system that includes aD and dD, is a normal **aKDdKDad∅**-system.

Let us consider some properties of this system.

4.6.1 The alethic-deontic octagon

Figure 7 displays the alethic-deontic octagon in the system **aKDdKDad∅**. The octagon is interpreted as usual.

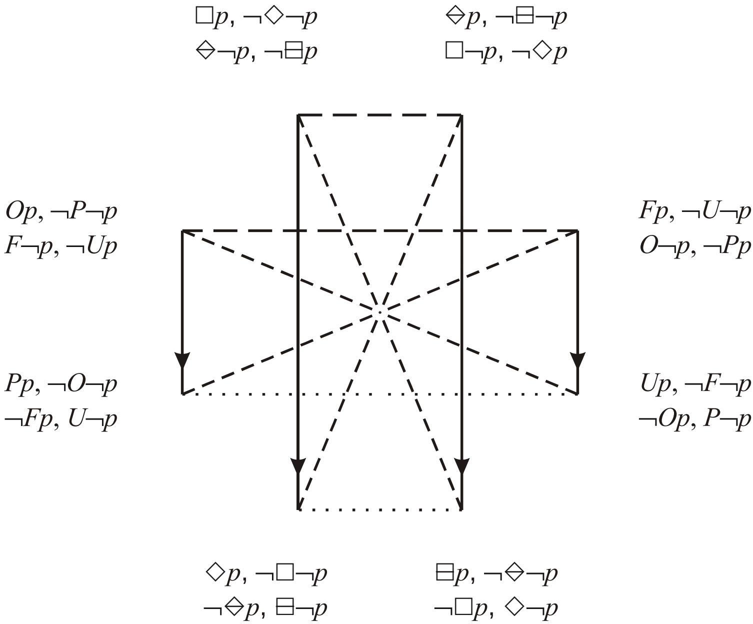

Figure 7. The Alethic-Deontic Octagon, **MADL** + {aD, dD} **(S5).**

Note that the figure essentially is a combination of the ad octagon for the system **aKDdKad∅** and the ad octagon for the system **aKdKDad∅**. Now, this should come as no surprise, since **aKDdKDad∅** includes every sentence in **aKDdKad∅** and in **aKdKDad∅**. Furthermore, since **MADL** + {aD, dD} contains these systems, it is a **aKDdKad∅**-system, as well as a **aKdKDad∅**- and a **aKDdKDad∅**-system. No mixed axioms are included in the system. Hence, no interesting relationships between deontic and alethic propositions are forthcoming.

The next system we consider includes both a pure additional alethic axiom and a mixed axiom.

4.7 aKDdKadMO

The smallest normal alethic-deontic logic that includes the axiom aD and MO, i.e. the sentences □p → ◇p and □p → Op, is **aKDdKadMO**. Accordingly, **aKDdKadMO** = **MADL** + {aD, MO}. We will also call this system **S6**. Since it is a normal alethic-deontic system **aKDdKadMO** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. A normal **aKDdKadMO**-system is any normal alethic-deontic system that includes aD and MO, or in other words, any normal alethic-deontic system that is an extension of **aKDdKadMO**.

Let us consider the alethic-deontic octagon in **aKDdKadMO**.

4.7.1 The alethic-deontic octagon

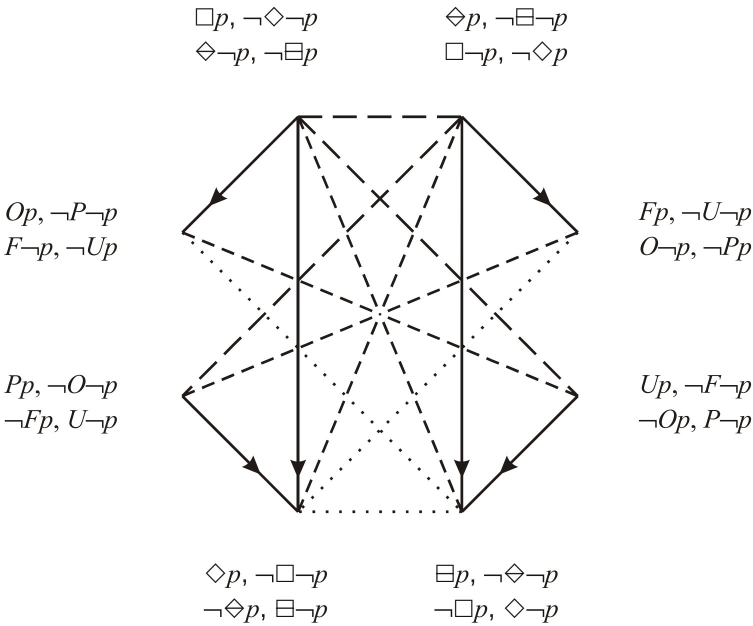

Figure 8. The Alethic-Deontic Octagon, MADL + {aD, MO} **(S6).**

4.8 SADL, aKdKadOCMO

aKdKadOCMO is the smallest normal alethic-deontic logic that includes the axioms OC and MO, i.e. the sentences $Op \rightarrow \Diamond p$ and $\Box p \rightarrow Op$. Accordingly, **aKdKadOCMO** = **MADL** + {OC, MO}. We will also call this system **S8** or Standard alethic-deontic logic **(SADL)**. Since it is a normal alethic-deontic system **aKdKadOCMO** includes PL, the axioms aK and dK, the usual definitions of the alethic and deontic operators, modus ponens, □-necessitation and O-necessitation. A normal **aKdKadOCMO**-system is any normal alethic-deontic system that includes OC and MO, or in other words, any normal alethic-deontic system that is an extension of **aKdKadOCMO**.

Let us consider some properties of this system.

4.8.1 The alethic-deontic octagon

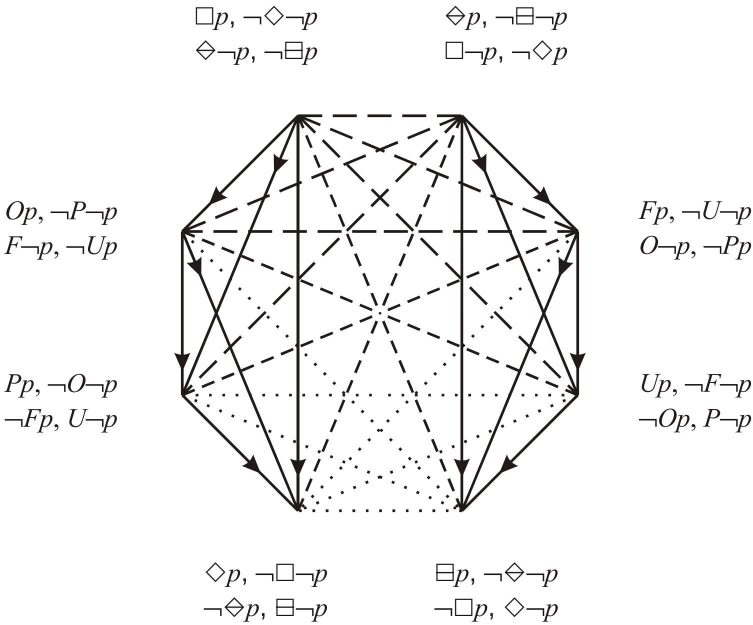

Figure 9. The Alethic-Deontic Octagon, aKdKadOCMO, SADL (S8).

4.8.2 Deductively equivalent systems

It is easy to see that all systems above are included in **aKdKadOCMO**. **MADL** is included since it is included in every normal ad system. Every extension of minimal alethic-deontic logic discussed so far in this paper is constructed by adding one or several of the axioms aD, dD, OC and MO to this system. The sentences aD and dD are theorems already in **aKdKadOC**. So, it is obvious that these sentences are provable also in **aKdKadOCMO**.

Hence, **aKdKadOCMO** includes **aKDdKad∅**, **aKdKDad∅** and **aKDdKDad∅**. Furthermore, it is also obvious that **aKdKadOC**, **aKdKadMO**, and **aKDdKadMO** are included in **aKdKadOCMO** since dD, OC and MO are theorems in **aKdKadOCMO**. Section 5 includes information about the relationships between all logics mentioned in this essay.

This completes our discussion of various alethic-deontic systems in this paper. We will end this article with some information about how the systems in this essay are related to each other.

5. Relationships between systems

Figure 10 displays the relationships between the systems we have discussed in this paper. Systems higher up in the diagram are stronger than systems lower down. So, S8 is the strongest system and S1 the weakest system. All other systems are included in S8 and S1 is included in all other systems.

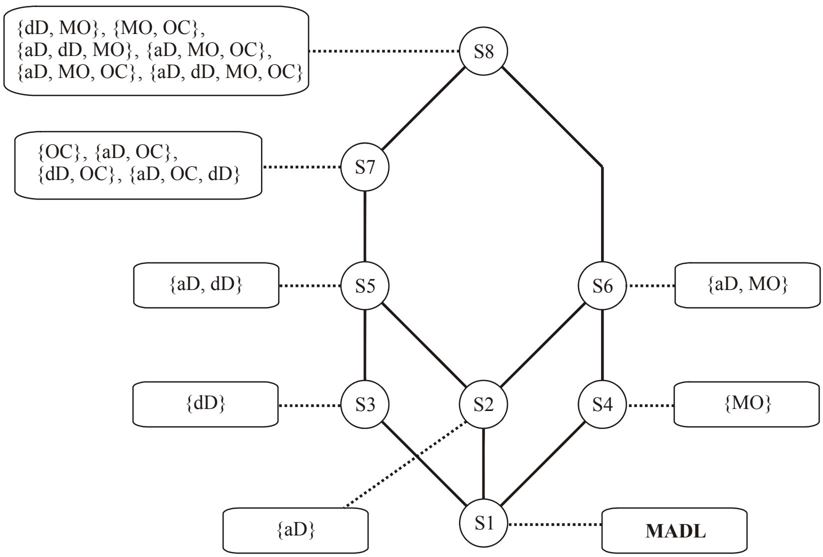

Figure 10. Relationships between some ad systems.

Nr	Systematic name	Extensions of MADL	Equivalent Systems
S1	**aKdKad∅**	**MADL**	
S2	**aKDdKad∅**	**MADL** + {aD}	
S3	**aKdKDad∅**	**MADL** + {dD}	
S4	**aKdKadMO**	**MADL** + {MO}	
S5	**aKDdKDad∅**	**MADL** + {aD, dD}	
S6	**aKDdKadMO**	**MADL** + {aD, MO}	
S7	**aKdKadOC**	**MADL** + {OC}	**MADL** + {aD, OC}**, MADL** + {dD, OC}, **MADL** + {aD, dD, OC}
S8	**aKdKadOCMO**	**MADL** + {OC, MO}	**MADL** + {aD, MO}**, MADL** + {aD, dD, MO}**, MADL** + {aD, OC, MO}**,** **MADL** + {aD, OC, MO}**, MADL** + {aD, dD, OC, MO}

Comment 34. In this paper I have described a set of alethic-deontic systems that include alethic and deontic operators that are used to symbolize various deontic and alethic modal concepts. But all systems have many possible informal readings. In Rönnedal (2012) I mention some interpretations of various bimodal systems. If we interpret □ as an epistemic operator and O as a doxastic operator, we obtain a set of epistemic-doxastic systems. If □ is read as "It is always the case that" or "It is and it is always going to be the case that" and O as "It is always going to be the case that", we obtain a set of bimodal temporal systems, etc. So, the results in this paper should be interesting not only to alethic-deontic logicians, but to any logician who wants to develop some kind of bimodal system.

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