Alethic-Deontic Logic: Deontic Accessibility Defined in Terms of Alethic Accessibility

Daniel Rönnedal

Abstract
According to many normative theories, to say that something ought to be, or ought to be done, is to state that the being or doing of this thing is in some sense a necessary condition (requirement) of something else. In this paper, I explore the consequences of such a view. I consider what kind of alethic-deontic logic is appropriate for theories of this sort. Alethic-deontic logic is a kind of bimodal logic that combines ordinary alethic (modal) logic and deontic logic. Ordinary alethic logic is a branch of logic that deals with modal concepts, such as necessity and possibility, modal sentences, arguments and systems. Deontic logic is the logic of norms. It deals with normative words, such as "ought", "right" and "wrong", normative sentences, arguments and systems. I will define the so-called deontic accessibility relation in terms of the so-called alethic accessibility relation, and I will examine the consequences of this definition. It will turn out that a particular alethic-deontic system, *Strong alethic-deontic logic*, is plausible given this definition. By adding a certain frame-condition, *the accessibility condition*, we obtain a slightly stronger system, *Full alethic-deontic logic*. Some of the technical details of these systems are briefly described. Most of the systems mentioned in this paper are developed in more detail elsewhere.

1. Introduction

Georg Henrik von Wright has suggested that "[t]o say that something ought to be, or ought to be done, is to state that the being or doing of this thing is a necessary condition (requirement) of something else" (von Wright (1971, p. 161)). He goes on:

> [T]o say that something ought to be or ought to be done is to say that the being or doing of this thing is a necessary condition of a certain other thing which is taken for granted or presupposed in the context. An 'ought'-statement is typically an *elliptic* statement of a necessary requirement. … This suggestion seems to me, on the whole, acceptable. If we accept it, then we are always, when confronted with

> an 'ought', entitled to raise the question 'Why?', *i.e.* to ask for the thing for which this or that is alleged to be a necessary requirement. (von Wright (1971 pp. 171–172))

Other philosophers have given similar analyses of some fundamental normative concepts. According to Alan Ross Anderson:

> The intimate connection between obligations and sanctions in normative systems suggests that we might profitably begin by considering some penalty or sanction *S*, and *define* the deontic or normative notions of obligation, etc. along the following lines: a state-of-affairs *p* is obligatory if the falsity of *p* entails the sanction *S*; *p* is forbidden if *p* entails the sanction *S*; and *p* is permitted if it is possible that *p* is true without the sanction *S* being true. (Anderson (1956. p. 170))

By adding these definitions to various systems of alethic modal logic, Anderson achieves a kind of "reduction" of monadic deontic logic to alethic modal logic. A similar analysis is offered by Stig Kanger (1957). The basic idea is that it ought to be the case that A iff A is a necessary condition for avoiding the sanction or for meeting some kind of demands (e.g. the demands of morality). (See also Anderson (1956), (1958), (1959), (1967) and Åqvist (1987), Chapter IV.)

In this paper, I will explore a set of normative theories that in some sense share this basic idea and consider what kind of alethic-deontic logic is appropriate for systems of this kind. Even though such systems are similar to those developed by von Wright, Anderson and Kanger, they differ from the latter in several important ways. Roughly, according to the theories we will focus on in this paper:

> It ought to be the case that A iff A is a necessary condition for creating (obtaining) a possible world that has property M, where M can be almost any property in which we are interested.

The possible world w can, for instance, have M iff w is good enough, meets the requirements of morality, is morally acceptable, has a total amount of value that is positive, above a certain threshold or maximal, is at least as good as every other (alethically accessible) world, doesn't contain any violations of rights, or is a Kingdom of Ends, etc. According to a theory of this kind, one ought to perform an action iff performing this action is a necessary condition for creating (obtaining) a possible world that has property M. In other words, one ought to perform an action iff the state of affairs that consists in one's performing this action is a necessary condition for creating (obtaining) a possible world that has the property M. Or again, one ought to perform this action iff the state of affairs that consists in one's performing this action obtains in every possible world that has property M. More precisely, all the theories we focus on in this paper define our basic deontic concepts in the following way:

> "It ought to be the case that A" is true in the possible world w iff "A" is true in every possible world that is alethically accessible from w and that has property M.
>
> "It is permitted that A" is true in the possible world w iff "A" is true in at least one possible world that is alethically accessible from w and that has property M.
>
> "It is forbidden that A" is true in the possible world w iff "not-A" is true in every possible world that is alethically accessible from w and that has property M.

Almost every, and perhaps every plausible theory taking this form – and "defining" the alethic accessibility relation in the same way – has the same alethic-deontic logic, even though "M" may stand for many different properties. An important subclass of theories of this kind is "doing the best we can" theories. The basic idea behind these theories is that we ought to do our best, or that we ought to do the best we can. One theory of this kind has, for instance, been developed by Fred Feldman (see Feldman (1986)). According to Feldman, "all of our moral obligations boil down to one - we morally ought to do the best we can." And by this he means, "we morally ought to do what we do in the intrinsically best possible worlds still accessible to us" (Feldman (1986, xi)). He goes on to say: "As I see it… what a person ought to do as of a time is what he does in the intrinsically best worlds accessible to him as of that time" (Feldman (1986, p. 13)). According to a theory of this kind, we can, for instance, define the concept of ought in the following way:

> "It ought to be the case that A" is true in the possible world w iff "A" is true in every possible world w′ that is alethically accessible from w and that is such that there is no other possible world w″ that is alethically accessible from w that is better than w′.

This idea can (in principle) be combined with almost any value-theory and with almost any analysis of the relation "better than". Intuitively, the definition entails that one ought to do A iff one does A in all the best alethically accessible worlds. If some kind of hedonism is true, then the possible world w is better than the possible world w′ iff the total amount of well-being ("pleasure" over "pain") is higher in w than in w′. If something else has value, e.g. justice, freedom, virtue, knowledge, beauty, friendship, love etc., these values will influence the relative values of different possible worlds. We will not develop on this here. The important thing to note is that many normative theories seem to share the same basic, formal structure. We therefore have good reason to question what sort of alethic-deontic logic is appropriate for theories of this kind.

Alethic-deontic logic is a form of bimodal logic that combines ordinary alethic (modal) logic and deontic logic. Ordinary alethic logic is a branch of logic that deals with modal concepts, such as necessity and possibility, modal sentences, arguments and systems. For some introductions, see e.g. Chellas (1980), Blackburn, de Rijke & Venema (2001), Blackburn, van Benthem & Wolter (eds.) (2007), Fitting & Mendelsohn (1998), Gabbay (1976), Gabbay & Guenthner (2001), Kracht (1999), Garson (2006), Girle (2000), Lewis & Langford (1932), Popkorn (1994), Segerberg (1971), and Zeman (1973). Deontic logic is the logic of norms. It deals with normative words, such as "ought", "right" and "wrong", normative sentences, arguments and systems. Introductions to this branch of logic can be found in e.g. Gabbay, Horty, Parent, van der Meyden & van der Torre (eds.) (2013), Hilpinen (1971), (1981), Rönnedal (2010), and Åqvist (1987), (2002). Alethic-deontic logic contains both modal and normative concepts and can be used to study how the two interact. In the paper Rönnedal (2012) I say more about various bimodal systems and in Rönnedal (2015) I prove some interesting theorems in some alethic-deontic systems (see also Rönnedal (2012b) and (2015b)). Anderson was perhaps the first philosopher to combine alethic and deontic logic (see Anderson (1956)). Fine & Schurz (1996), Gabbay & Guenthner (2001), Gabbay, Kurucz, Wolter & Zakharyaschev (2003), Kracht (1999), and Kracht & Wolter (1991) offer more information about how to combine various logical systems.

In monadic deontic logic the truth-conditions for normative sentences are usually defined in terms of a primitive deontic accessibility relation. The truth-conditions for "obligation-sentences", for instance, are often defined in the following way: "It ought to be the case that A" is true in a possible world w iff "A" is true in every possible world that is deontically accessible from w.

In Rönnedal (2012) I use two primitive accessibility relations, one alethic and one deontic. In this paper, we will define the deontic accessibility relation in terms of the alethic accessibility relation and see what follows. According to this definition, the possible world w′ is deontically accessible from the possible world w iff w′ is alethically accessible from w and w′ has the property M. Given this definition of the deontic accessibility relation, it follows from the standard definition of the truth-conditions for "ought-sentences" that "it ought to be the case that A" is true in the possible world w iff "A" is true in every possible world that is alethically accessible from w and that has property M.

In this paper I only consider some alethic-deontic systems. I don't say anything about how various norms might be related to different moments in time. However, all the systems I describe can be embedded in a temporal dimension in a more or less straightforward way. For an idea about how this might be possible, see Rönnedal (2012c) (see also Rönnedal (2012b)).

The essay is divided into seven sections. In part 2 I describe the syntax of our systems and in part 3 I talk about their semantics. Part 4 deals with the proof theoretic characterization of our logics, while part 5 offers some examples of theorems in the various systems and an analysis of some arguments. Part 6 gives information about some deductively equivalent systems; and Part 7 details soundness and completeness theorems.

2. Syntax

Alphabet. (i) A denumerably infinite set Prop of proposition letters p, q, r, s, t, p_1, q_1, r_1, s_1, t_1, p_2, q_2, r_2, s_2, t_2…, (ii) the primitive truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (material implication), and $\equiv$ (material equivalence), (iii) the modal (alethic) operators $\Box$, $\Diamond$, and $\Diamond\!\!\!\!-$, (iv) the deontic operators O, P, and F, and (v) the brackets (,).

Language. The language L is the set of well-formed formulas (wffs) generated by the usual clauses for proposition letters and propositionally compound sentences, and the following clauses: (i) if A is a wff, then $\Box$A, $\Diamond$A and $\Diamond\!\!\!\!-$A are wffs, (ii) if A is a wff, then OA, PA and FA are wffs, and (iii) nothing else is a wff.

Definitions. KA = PA $\wedge$ P$\neg$A, and NA = (OA $\vee$ O$\neg$A). $\bot$ (falsum) and $\top$ (verum) are defined as usual.

Capital letters A, B, C … are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter Γ represents an arbitrary set of formulas. Outer brackets around sentences are usually dropped if the result is not ambiguous. We also use a, b, c, … as proposition letters.

The translationfunction t. To understand the intended interpretation of the formal language in this essay we can use the following translation function. $t(\neg A)$ = It is not the case that $t(A)$. $t(A \supset B)$ = If $t(A)$, then $t(B)$. And similarly for all other propositional connectives. $t(\Box A)$ = It is necessary that $t(A)$. $t(\Diamond A)$ = It is possible that $t(A)$. t(⟠A) = It is impossible that $t(A)$. $t(OA)$ = It ought to be the case that (it is obligatory that) $t(A)$. $t(PA)$ = It is permitted that $t(A)$. $t(FA)$ = It is forbidden that $t(A)$. $t(KA)$ = It is optional (deontically contingent) that $t(A)$. $t(NA)$ = It is non-optional (deontically non-contingent) that $t(A)$. If $t(p)$ and $t(q)$ are English sentences, we can use t to translate a formal sentence containing p and q into English. For instance, let $t(p)$ be "You are honest" and $t(q)$ be "You lie". Then the t-translation of "$(Op \land \Box(p \supset \neg q)) \supset O\neg q$" is "If it ought to be the case that you are honest and it is necessary that if you are honest then it is not the case that you lie, then it ought to be the case that it is not the case that you lie".[1] This is an instance of the so-called means-end principle that says that every necessary consequence of what ought to be ought to be.

There seem to be several different kinds of necessity and possibility: logical, metaphysical, natural, historical etc. If not otherwise stated, we will usually mean "historical necessity" by "necessity" in this paper.

3. Semantics

3.1 Basic concepts

Alethic-deontic frame. An (alethic-deontic) frame F is a relational structure $\langle W, R, S \rangle$, where W is a non-empty set of possible worlds, and R and S are two binary accessibility relations on W.

R "corresponds" to the operators □, ◇ and ⟠, and S to the operators O, P and F. If Rww', we shall say that w' is R-accessible or alethically accessible from w, and if Sww', that w' is S-accessible or deontically accessible from w.

Alethic-deontic model. An (alethic-deontic) model M is a pair $\langle F, V \rangle$ where: (i) F is an alethic-deontic frame; and (ii) V is a valuation or interpretation function, which assigns a truth-value T (true) or F (false) to every proposition letter p in each world $w \in W$.

[1] Of course, stylistically this is not a particularly "nice" sentence. Nevertheless, it makes a good job in conveying the informal meaning of the formal sentence.

When M = <F, V> we say that M is *based on* the frame F, or that F is the frame *underlying* M. To save space, we shall also use the following notation for a model: <W, R, S, V>, where W, R, S and V are interpreted as usual. "**F**" stands for a class of frames and "**M**" for a class of models.

Truth in a model. Let M be any model <F, V>, based on a frame F = <W, R, S>. Let w be any member of W and let A be a well-formed sentence in L. $\Vdash_{M, w}$ A abbreviates *A is true at or in the possible world w in the model M*. The truth conditions for proposition letters and sentences built by truth-functional connectives are the usual ones. The truth conditions for the remaining sentences in L are given by the following clauses: (i) $\Vdash_{M, w} \Box A$ iff for all w′ ∈ W such that Rww′: $\Vdash_{M, w'}$ A, (ii) $\Vdash_{M, w} \Diamond A$ iff for at least one w′ ∈ W such that Rww′: $\Vdash_{M, w'}$ A, (iii) $\Vdash_{M, w}$ ⟠A iff for all w′ ∈ W such that Rww′: $\Vdash_{M, w'} \neg A$, (iv) $\Vdash_{M, w}$ OA iff for all w′ ∈ W such that Sww′: $\Vdash_{M, w'}$ A, (v) $\Vdash_{M, w}$ PA iff for at least one w′ ∈ W such that Sww′: $\Vdash_{M, w'}$ A, and (vi) $\Vdash_{M, w}$ FA iff for all w′ ∈ W such that Sww′: $\Vdash_{M, w'} \neg A$.

Validity. A sentence A is valid on or in a class of frames **F** ($\Vdash_{\mathbf{F}}$ A) iff A is true at every world in every model based on some frame in this class.

Satisfiability. A set of sentences Γ is satisfiable in a class of frames **F** iff at some world in some model based on some frame in **F** every sentence in Γ is true. Γ is satisfiable in a model iff at some possible world in the model all sentences in Γ are true.

Logical consequence. A sentence B is a logical consequence of a set of sentences Γ on or in a class of frames **F** ($\Gamma \Vdash_{\mathbf{F}}$ B) iff B is true at every world in every model based on a frame in **F** at which all members of Γ are true.

3.2 Conditions on a frame

We will begin this section with exploring several different conditions on our frames. These conditions are divided into three classes. The first class tells us something about the formal properties of the relation R, the second about the formal properties of the relation S, and the third about how S and R are related to each other in a frame. Then we will go one and define the deontic accessibility relation in terms of the alethic accessibility relation and consider the consequences of this definition.

The variables 'x', 'y', 'z' and 'w' in tables 1, 2 and 3 are taken to range over possible worlds in W, and the symbols ∧, ⊃, ∀ and ∃ are used as metalogical symbols in the standard way. Let F = <W, R, S> be a bimodal frame and M = <W, R, S, V> be a bimodal model. If S is serial in W, i.e. if ∀x∃ySxy, we say that S satisfies or fulfils condition C-dD and also that F and

M satisfy or fulfil condition C-dD and similarly in all other cases. C-dD is called "C-dD" because the tableau rule T-dD "corresponds" to C-dD and the sentence dD is valid on the class of all frames that satisfies condition C-dD and similarly in all other cases. Let C be any of the conditions in table 1, 2 or 3. Then a C-frame is a frame that satisfies condition C and a C-model is a model that satisfies C.

3.2.1 Conditions on the relation R

Condition	Formalization of Condition
C-aT	$\forall xRxx$
C-aD	$\forall x\exists yRxy$
C-aB	$\forall x\forall y(Rxy \supset Ryx)$
C-a4	$\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$
C-a5	$\forall x\forall y\forall z((Rxy \wedge Rxz) \supset Ryz)$

Table 1

3.2.2 Conditions on the relation S

Condition	Formalization of Condition
C-dD	$\forall x\exists ySxy$
C-d4	$\forall x\forall y\forall z((Sxy \wedge Syz) \supset Sxz)$
C-d5	$\forall x\forall y\forall z((Sxy \wedge Sxz) \supset Syz)$
C-dT′	$\forall x\forall y(Sxy \supset Syy)$
C-dB′	$\forall x\forall y\forall z((Sxy \wedge Syz) \supset Szy)$

Table 2

3.2.3 Mixed conditions on alethic-deontic frames

Condition	Formalization of Condition
C-MO	$\forall x\forall y(Sxy \supset Rxy)$
C-OC	$\forall x\exists y(Sxy \wedge Rxy)$
C-OC′	$\forall x\forall y(Sxy \supset \exists z(Ryz \wedge Syz))$
C-MO′	$\forall x\forall y\forall z((Sxy \wedge Syz) \supset Ryz)$
C-ad4	$\forall x\forall y\forall z((Rxy \wedge Syz) \supset Sxz)$
C-ad5	$\forall x\forall y\forall z((Rxy \wedge Sxz) \supset Syz)$
C-PMP	$\forall x\forall y\forall z((Sxy \wedge Rxz) \supset \exists w(Ryw \wedge Szw))$
C-OMP	$\forall x\forall y\forall z((Rxy \wedge Syz) \supset \exists w(Sxw \wedge Rwz))$
C-MOP	$\forall x\forall y\forall z((Sxy \wedge Ryz) \supset \exists w(Rxw \wedge Swz))$

Table 3

3.3 Definition of the deontic accessibility relation in terms of the alethic accessibility relation

Rönnedal (2012) gives information about some of the relationships between the conditions introduced above. The appendix in Rönnedal (2012b) offers more information. In this section we will see what happens if we define the deontic accessibility relation in terms of the alethic accessibility relation in a certain way. Here is our definition:

> Def(S) ∀x∀y(Sxy ≡ (Rxy ∧ My)). The possible world y is deontically accessible from x iff y is alethically accessible from x and y has the property M.

In our theorems below we treat M as an ordinary monadic predicate. But it can be replaced by almost any predicate and the proofs will go through anyway. It follows that, as we mentioned in the introduction, almost every, and perhaps every plausible theory taking this form – and "defining" the alethic accessibility relation in the same way – has the same alethic-deontic logic, even though "M" may stand for many different properties. As we also mentioned in the introduction, an important subclass of theories of this kind is "doing the best we can" theories. According to these theories, we ought to do our best, or the best we can (see the introduction). For theories of this kind, we can replace "My" in Def(S) by "¬∃z((¬z=y ∧ Rxz) ∧ Bzy)", where Bzy is read "z is better than y". According to these theories, the deontic accessibility relation is defined in the following way: ∀x∀y(Sxy ≡ (Rxy ∧ ¬∃z((¬z=y ∧ Rxz) ∧ Bzy))), which says that the possible world y is deontically accessible from the possible world x iff y is alethically accessible from x and there is no other possible world z alethically accessible from x that is better than y.

Before we introduce our theorems, we will consider one more frame- and model-condition.

> C-adD ∀x∃y(Rxy ∧ My)

According to this condition, every possible world x can see at least one possible world y that has the property M. We will also call C-adD *the accessibility condition*.

We are now in a position to establish some theorems that tell us something about the consequences of Def(S).

Theorem 1. R is an equivalence relation (i) iff R is reflexive (C-aT), symmetric (C-aB) and transitive (C-a4), (ii) iff R is reflexive (C-aT) and Euclidean (C-a5), (iii) iff R is serial (C-aD), symmetric (C-aB) and transitive (C-a4), (iv) iff R is serial (C-aD), symmetric (C-aB) and Euclidean (C-a5).

Proof. Straightforward.

It is reasonable to assume that the alethic accessibility relation is an equivalence relation given almost any interpretation of our alethic concepts, for instance if we think about necessity, possibility and impossibility as historical, nomological, metaphysical or logical. If we assume this, our alethic operators will behave as S5-operators.

Theorem 2. (i) Def(S) and C-adD entail C-dD and C-OC. (ii) Def(S) entails C-MO and C-MO′. (iii) Def(S) and C-aT entail C-dT′ and C-OC′. (iv) Def(S) and C-aB entail C-dB′. (v) Def(S) and C-a4 entail C-d4 and C-ad4. (vi) Def(S), C-aB and C-a4 entail C-d5 and C-ad5. (vii) Def(S), C-aT and C-a4 entail C-OMP. (viii) Def(S), C-aT, C-aB and C-a4 entail C-PMP.

Proof. Left to the reader.

Theorem 3. (i) If Def(S) is true and R is an equivalence relation in a model M, then M satisfies C-d4, C-d5, C-dT′, C-dB′, C-OC′, C-MO, C-MO′, C-ad4, C-ad5, C-PMP and C-OMP, but not necessarily C-dD, C-OC and C-MOP. (ii) If we add the condition C-adD $\forall x \exists y(Rxy \wedge My)$ (i.e. for every world x there is a world y that is alethically accessible from x and that has property M), then M also satisfies C-dD and C-OC (but not necessarily C-MOP).

Proof. This follows from theorem 1 and theorem 2.

3.4 Classification of frame classes and the logic of a class of frames

The conditions on our frames listed in tables 1, 2 and 3 can be used to obtain a categorization of the set of all frames into various kinds. We shall say that **F**(C_1, …, C_n) is the class of (all) frames that satisfies the conditions C_1, …, C_n. E.g. **F**(C-dD, C-aT, C-MO) is the class of all frames that satisfies C-dD, C-aT and C-MO. $\mathbf{F}_s$ is the set of all frames where the deontic accessibility relation is defined in terms of the alethic accessibility relation, i.e. that satisfies Def(S); and an $\mathbf{F}_s$-frame is a frame that satisfies Def(S). $\mathbf{F}_s$(Eq) is the class of all $\mathbf{F}_s$-frames where R is an equivalence relation; and $\mathbf{F}_s$(Eq, C-adD) or $\mathbf{F}_s$(Eq, adD) is the class of all $\mathbf{F}_s$-frames that satisfies C-adD (and where R is an equivalence relation).

The set of all sentences (in L) that are valid in a class of frames **F** is called the logical system of (the system of or the logic of) **F**, in symbols S(**F**) = {A

∈ L: $\Vdash_{\mathbf{F}}$ A}. E.g. S(**F**(C-dD, C-aT, C-MO)) is the set of all sentences that are valid on the class of all frames that satisfies C-dD, C-aT and C-MO.

By using this classification of frame classes we can define a large set of systems. In the next section we will develop semantic tableau systems that exactly correspond to these logics. We will see that $\mathbf{F}_s$(Eq) corresponds to Strong alethic-deontic logic and $\mathbf{F}_s$(Eq, C-adD) to Full alethic-deontic logic.

4. Proof theory

4.1 Semantic tableaux

We use a kind of indexed semantic tableau systems in this paper. A similar technical apparatus can be found in e.g. Priest (2008). The propositional part is basically the same as in Smullyan (1968) and Jeffrey (1967).

The concepts of semantic tableau, branch, open and closed branch etc. are defined as in Priest (2008) and Rönnedal (2012b, p. 131). For more on semantic tableaux, see D'Agostino, Gabbay, Hähnle & Posegga (1999), Fitting (1983), and Fitting & Mendelsohn (1998).

4.2 Tableau rules

4.2.1 Propositional rules

We use the same propositional rules as in Priest (2008) and Rönnedal (2012b). These rules are interpreted exactly as in monomodal systems.

4.2.2 Basic a-Rules

□	◇	◇̶
□A, i irj ↓ A, j	◇A, i ↓ irj A, j where j is new	◇̶A, i ↓ □¬A, i
¬□	¬◇	◇̶
¬□A, i ↓ ◇¬A, i	¬◇A, i ↓ □¬A, i	¬◇̶A, i ↓ ◇A, i

Table 4

4.2.3 Basic d-Rules

The basic d-Rules look exactly like the basic a-Rules, except that □ is replaced by O, ◇ by P, ⬗ by F, and r by s. We give them similar names.

4.2.4 Accessibility rules (a-Rules)

T-aD	T-aT	T-aB	T-a4	T-a5
i	i	irj	irj	irj
↓	↓	↓	jrk	irk
irj	iri	jri	↓	↓
where j is new			irk	jrk

Table 5

4.2.5 Accessibility rules (d-Rules)

T-dD	T-d4	T-d5	T-dT′	T-dB′
i	isj	isj	isj	isj
↓	jsk	isk	↓	jsk
isj	↓	↓	jsj	↓
where j is new	isk	jsk		ksj

Table 6

4.2.6 Accessibility rules (ad-Rules)

T-MO	T-MO′	T-OC	T-OC′	
isj	isj	i	isj	
↓	jsk	↓	↓	
irj	↓	isj	jrk	
	jrk	irj	jsk	
		where j is new	where k is new	
T-ad4	**T-ad5**	**T-PMP**	**T-OMP**	**T-MOP**
irj	irj	isj	irj	isj
jsk	isk	irk	jsk	jrk
↓	↓	↓	↓	↓
isk	jsk	jrl	isl	irl
		ksl	lrk	lsk
		where l is new	where l is new	where l is new

Table 7

4.3 Tableau systems

A tableau system is a set of tableau rules. A (normal) alethic-deontic tableau system includes all propositional rules and all basic a- and d-Rules (sections 4.2.1 to 4.2.3 and table 4). The minimal (normal) bimodal tableau system is called “T”. By adding any subset of the accessibility rules introduced in sections 4.2.4 to 4.2.6 (tables 5, 6 and 7), we obtain an extension of T. Some of these are deductively equivalent, i.e. contain exactly the same set of theorems. We use the following conventions for naming systems. We write “$aA_1 \ldots A_n dB_1 \ldots B_n adC_1 \ldots C_n$”, where $A_1 \ldots A_n$ is a list (possibly empty) of (non-basic) a-Rules, $B_1 \ldots B_n$ is a list (possibly empty) of (non-basic) d-Rules, and $C_1 \ldots C_n$ is a list (possibly empty) of (non-basic) ad-Rules. We abbreviate by omitting the initial “a” in the names of the a-Rules after the first occurrence and similarly for the d- and ad-Rules. Also, the initial “T-” in every rule is usually omitted. If a system doesn’t include any (non-basic) a-Rules, we may also omit the initial “a”. The same goes for systems with no (non-basic) d- or ad-Rules. We will sometimes add “TS-” in the beginning of a name of a system to indicate that it is a tableau system we are talking about.

E.g. aDTB45dD45T′B′adOCMOOC′MO′45PMPOMP is the normal, alethic-deontic tableau system that includes the rules T-aD, T-aT, T-aB, T-a4, T-a5, T-dD, T-d4, T-d5, T-dT′, T-dB′, T-OC, T-MO, T-OC′, T-MO′, T-ad4, T-ad5, T-PMP and T-OMP. This system, which includes several redundant rules, will also be called T_s(Eq, adD) (since it corresponds to $\mathbf{F}_s$(Eq, adD)) or *Full alethic-deontic logic* (FADL). If we drop T-OC, and T-dD from this system, we obtain a system we will call T_s(Eq) (since it corresponds to $\mathbf{F}_s$(Eq)) or *Strong alethic-deontic logic* (StADL). There are many different systems that are equivalent to FADL and StADL (see section 6).

4.4 Some proof theoretical concepts and the logic of a tableau system

The concepts of proof, theorem, derivation, consistency, inconsistency in a system etc. can be defined in the usual way. $\vdash_S A$ says that A is a theorem in the system S and $\Gamma \vdash_S A$ says that A is derivable from Γ in S.

Let S be a tableau system. Then the logic (or the (logical) system) of S, L(S), is the set of all sentences (in L) that are provable in S, in symbols L(S) = $\{A \in L: \vdash_S A\}$. E.g. L(aTdDadMO) is the set of all sentences that are provable in the system aTdDadMO, i.e. in the system that includes the basic rules and the (non-basic) rules T-aT, T-dD and T-MO.

5. Examples of theorems and arguments

5.1 Examples of theorems

Theorem 4. The sentences in tables 8 to 16 are theorems (or more precisely theorem schemas) in the indicated systems.

Proof. Left to the reader.

Name	Theorem	System
aK	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	T
aT	$\Box A \supset A$	TS-aT
aD	$\Box A \supset \Diamond A$	TS-aD
aB	$A \supset \Box\Diamond A$	TS-aB
a4	$\Box A \supset \Box\Box A$	TS-a4
a5	$\Diamond A \supset \Box\Diamond A$	TS-a5

Table 8

Name	Theorem	System
dK	$O(A \supset B) \supset (OA \supset OB)$	T
dD	$OA \supset PA$	TS-dD
d4	$OA \supset OOA$	TS-d4
d5	$PA \supset OPA$	TS-d5
dT′	$O(OA \supset A)$	TS-dT′
dB′	$O(POA \supset A)$	TS-dB′

Table 9

Name	Theorem	System
MO	$\Box A \supset OA$	TS-MO
OC	$OA \supset \Diamond A$	TS-OC
OC′	$O(OA \supset \Diamond A)$	TS-OC′
MO′	$O(\Box A \supset OA)$	TS-MO′
ad4	$OA \supset \Box OA$	TS-ad4
ad5	$PA \supset \Box PA$	TS-ad5
PMP	$P\Box A \supset \Box PA$	TS-PMP
OMP	$O\Box A \supset \Box OA$	TS-OMP
MOP	$\Box OA \supset O\Box A$	TS-MOP

Table 10

Theorem	Sys	Theorem	Sys
FA ⊃ □FA	ad4	◇OA ⊃ OA	ad5
◇PA ⊃ PA	ad4	FA ∨ □PA	ad5
⟠PA ∨ PA	ad4	⟠OA ∨ OA	ad5

Table 11. Theorems in some systems (Sys = System)

Theorem	Sys	Theorem	Sys	Theorem	Sys
□A ⊃ PA	OC	⟠A ⊃ FA	MO	◇OA ⊃ O◇A	PMP
FA ⊃ ¬□A	OC	PA ⊃ ◇A	MO	P⟠A ⊃ ⟠OA	PMP
¬(OA ∧ ⟠A)	OC	¬(P¬A∧□A)	MO	◇PA ⊃ P◇A	OMP
¬(FA ∧ □A)	OC	¬(PA ∧ ⟠A)	MO	O⟠A ⊃ □FA	OMP
PA ∨ ◇¬A	OC	FA ∨ ◇A	MO	P◇A ⊃ ◇PA	MOP
P¬A ∨ ◇A	OC	OA ∨ ◇¬A	MO	□FA ⊃ O⟠A	MOP

Table 12. Theorems in some systems (Sys = System)

□(A ∧ B) ⊃ (OA ∧ OB)	(PA ∧ □(A ⊃ B)) ⊃ PB
(□A ∨ □B) ⊃ O(A ∨ B)	□(A ⊃ B) ⊃ (PA ⊃ PB)
(□A ∧ □B) ⊃ O(A ∧ B)	PA ⊃ (□(A ⊃ B) ⊃ PB)
P(A ∧ B) ⊃ (◇A ∧ ◇B)	(FB ∧ □(A ⊃ B)) ⊃ FA
P(A ∨ B) ⊃ (◇A ∨ ◇B)	□(A ⊃ B) ⊃ (FB ⊃ FA)
(PA ∨ PB) ⊃ ◇(A ∨ B)	FB ⊃ (□(A ⊃ B) ⊃ FA)
⟠(A ∨ B) ⊃ (FA ∧ FB)	(PA ∧ □(A ⊃ B)) ⊃ ◇B
(⟠A ∨ ⟠B) ⊃ F(A ∧ B)	□(A ⊃ B) ⊃ (PA ⊃ ◇B)
(⟠A ∧ ⟠B) ⊃ F(A ∨ B)	PA ⊃ (□(A ⊃ B) ⊃ ◇B)
□(A ≡ B) ⊃ (OA ≡ OB)	(KA ∧ □(A ⊃ B)) ⊃ PB
□(A ≡ B) ⊃ (PA ≡ PB)	□(A ⊃ B) ⊃ (KA ⊃ PB)
□(A ≡ B) ⊃ (FA ≡ FB)	KA ⊃ (□(A ⊃ B) ⊃ PB)
□(A ≡ B) ⊃ (¬OA ≡ ¬OB)	(KA ∧ □(A ⊃ B)) ⊃ ◇B
□(A ≡ B) ⊃ (KA ≡ KB)	□(A ⊃ B) ⊃ (KA ⊃ ◇B)
□(A ≡ B) ⊃ (NA ≡ NB)	KA ⊃ (□(A ⊃ B) ⊃ ◇B)
(OA ∧ □(A ⊃ B)) ⊃ OB	(¬OB ∧ □(A ⊃ B)) ⊃ ¬OA
□(A ⊃ B) ⊃ (OA ⊃ OB)	□(A ⊃ B) ⊃ (¬OB ⊃ ¬OA)
OA ⊃ (□(A ⊃ B) ⊃ OB)	¬OB ⊃ (□(A ⊃ B) ⊃ ¬OA)

Table 13. Theorems in TS-MO

□(A ⊃ B) ⊃ (□A ⊃ OB)
□(A ⊃ B) ⊃ (PA ⊃ ◇B)
□(A ⊃ B) ⊃ (⟠B ⊃ FA)

(O(A ∨ B) ∧ ⟠B) ⊃ OA
□((A ∨ B) ⊃ C) ⊃ ((OA ∨ OB) ⊃ OC)
□((A ∨ B) ⊃ C) ⊃ ((PA ∨ PB) ⊃ PC)
□((A ∨ B) ⊃ C) ⊃ (FC ⊃ (FA ∧ FB))
□(A ⊃ (B ∨ C)) ⊃ (PA ⊃ (PB ∨ PC))
□(A ⊃ (B ∨ C)) ⊃ ((FB ∧ FC) ⊃ FA)
□((A ∧ B) ⊃ C) ⊃ ((OA ∧ OB) ⊃ OC)
□(A ⊃ (B ∧ C)) ⊃ (OA ⊃ (OB ∧ OC))
□(A ⊃ (B ∧ C)) ⊃ (PA ⊃ (PB ∧ PC))
□(A ⊃ (B ∧ C)) ⊃ ((FB ∨ FC) ⊃ FA)
(O(A ∨ B) ∧ (□(A ⊃ C) ∧ □(B ⊃ C))) ⊃ OC
(O(A ∨ B) ∧ (□(A ⊃ C) ∧ □(B ⊃ D))) ⊃ O(C ∨ D)
(OA ∧ (□(A ⊃ B) ∧ □(A ⊃ C))) ⊃ (OB ∧ OC)
(O(A ∧ B) ∧ (□(A ⊃ C) ∨ □(B ⊃ D))) ⊃ O(C ∨ D)
(OA ∧ (□(A ⊃ B) ∨ □(A ⊃ C))) ⊃ O(B ∨ C)
(O(A ∧ B) ∧ (□(A ⊃ C) ∧ □(B ⊃ D))) ⊃ (OC ∧ OD)

Table 14. Theorems in TS-MO

(OA ∧ OB) ⊃ ◇(A ∧ B)	□(A ⊃ B) ⊃ (□A ⊃ PB)
(□A ∧ □B) ⊃ P(A ∧ B)	(□A ∧ □(A ⊃ B)) ⊃ PB
(OA ∨ OB) ⊃ ◇(A ∨ B)	□A ⊃ (□(A ⊃ B) ⊃ PB)
(□A ∨ □B) ⊃ P(A ∨ B)	□(A ⊃ B) ⊃ (FB ⊃ ¬□A)
O(A ∧ B) ⊃ (◇A ∧ ◇B)	(FB ∧ □(A ⊃ B)) ⊃ ¬□A
□(A ∧ B) ⊃ (PA ∧ PB)	FB ⊃ (□(A ⊃ B) ⊃ ¬□A)
□(A ⊃ B) ⊃ (OA ⊃ ◇B)	□(A ⊃ B) ⊃ (⟠B ⊃ ¬OA)
(OA ∧ □(A ⊃ B)) ⊃ ◇B	(⟠B ∧ □(A ⊃ B)) ⊃ ¬OA
OA ⊃ (□(A ⊃ B) ⊃ ◇B)	⟠B ⊃ (□(A ⊃ B) ⊃ ¬OA)

Table 15. Theorems in TS-OC

□(A ⊃ B) ⊃ (OA ⊃ PB)
(OA ∧ □(A ⊃ B)) ⊃ PB
□(A ⊃ B) ⊃ (FB ⊃ ¬OA)
(FB ∧ □(A ⊃ B)) ⊃ ¬OA
¬(O(A ∨ B) ∧ (⟠A ∧ ⟠B))
□((A ∨ B) ⊃ C) ⊃ ((OA ∨ OB) ⊃ PC)
□((A ∨ B) ⊃ C) ⊃ (FC ⊃ (¬OA ∧ ¬OB))
□(A ⊃ (B ∨ C)) ⊃ (OA ⊃ (PB ∨ PC))
□(A ⊃ (B ∨ C)) ⊃ ((FB ∧ FC) ⊃ ¬OA)

$\Box((A \wedge B) \supset C) \supset ((OA \wedge OB) \supset PC)$
$\Box((A \wedge B) \supset C) \supset (FC \supset (\neg OA \vee \neg OB))$
$\Box((A \wedge B) \supset C) \supset (FC \supset (P\neg A \vee P\neg B))$
$\Box(A \supset (B \wedge C)) \supset (OA \supset (PB \wedge PC))$
$\Box(A \supset (B \wedge C)) \supset ((FB \vee FC) \supset \neg OA)$
$\Box(A \supset (B \wedge C)) \supset ((\neg PB \vee \neg PC) \supset \neg OA)$
$(O(A \vee B) \wedge (\Box(A \supset C) \wedge \Box(B \supset C))) \supset PC$
$(O(A \vee B) \wedge (\Box(A \supset C) \wedge \Box(B \supset D))) \supset (PC \vee PD)$
$(OA \wedge (\Box(A \supset B) \wedge \Box(A \supset C))) \supset (PB \wedge PC)$
$(O(A \wedge B) \wedge (\Box(A \supset C) \vee \Box(B \supset D))) \supset (PC \vee PD)$
$(OA \wedge (\Box(A \supset B) \vee \Box(A \supset C))) \supset (PB \vee PC)$
$(O(A \wedge B) \wedge (\Box(A \supset C) \wedge \Box(B \supset D))) \supset (PC \wedge PD)$

Table 16. Theorems in TS-OC

Theorem 5. (i) All sentences in tables 8 – 16 except the “dD”, “OC” and “MOP”-sentences are theorems in Strong alethic-deontic logic (T_s(Eq)). (ii) All sentences in tables 8 – 16 except the “MOP”-sentences are theorems in Full alethic-deontic logic (T_s(Eq, adD)).

Proof. Left to the reader.

Theorem 6. (i) In Full alethic-deontic logic (T_s(Eq, adD)) the set of all sentences can be partitioned into the following, mutually exclusive, exhaustive subsets: $\Box A \wedge OA$, $OA \wedge \neg\Box A$, $PA \wedge P\neg A$, $FA \wedge \neg$⬲A, and $FA \wedge$ ⬲A. (ii) In Full alethic-deontic logic (T_s(Eq, adD)) the following is true: $\vdash \Box(A \equiv B) \supset (*A \equiv *B)$, where * = O, P, F, K and N. (iii) In Full alethic-deontic logic (T_s(Eq, adD)) there are exactly ten distinct modalities: A, ¬A, ◇A, □A, PA, OA, ¬◇A/⬲A, ¬□A, ¬PA/FA and ¬OA.

Proof. See Rönnedal (2015).

5.2 Examples of arguments

In this section we will illustrate how the systems we describe in this essay can be used to analyze some arguments formulated in English. Then we will show how we can prove that an argument is valid or invalid.

In every system that includes T-OC, $Op \supset \Diamond p$ is a theorem. This is one version of the so-called ought-implies-can principle (Kant’s law), which says that if it ought to be the case that p then it is possible that p, i.e. only something possible is obligatory. The contraposition of this theorem, ⬲p ⊃ ¬Op, is also provable. This theorem says that nothing impossible is obligatory.

Consider the following argument.

Argument 1
It is not possible that you stop and help this injured man and keep your promise to your friend.
Hence, it is not the case that you (all-things considered) ought to stop and help this injured man and that you (all-things considered) ought to keep your promise to your friend.

This argument seems valid, it seems impossible that the premise could be true and the conclusion false, or – in other words – that it is necessary that the conclusion is true if the premise is true. And, in fact, we can prove that it is (syntactically) valid in every alethic-deontic system that includes the tableau rule T-OC. Argument 1 can be formalized in our systems in the following way: $\neg\Diamond(h \wedge k) : \neg(Oh \wedge Ok)$, where h = You stop and help this man, and k = you keep your promise to your friend.

(1) $\neg\Diamond(h \wedge k)$, 0
(2) $\neg\neg(Oh \wedge Ok)$, 0
(3) $Oh \wedge Ok$, 0 [2, $\neg\neg$]
(4) Oh, 0 [3, $\wedge$]
(5) Ok, 0 [3, $\wedge$]
(6) $\Box\neg(h \wedge k)$, 0 [1, $\neg\Diamond$]
(7) 0s1 [T-OC]
(8) 0r1 [T-OC]
(9) h, 1 [4, 7, O]
(10) k, 1 [5, 7, O]
(11) $\neg(h \wedge k)$, 1 [6, 8, $\Box$]

↙ ↘

(12) $\neg h$, 1 [11, $\neg\wedge$]	(13) $\neg k$, 1 [11, $\neg\wedge$]
(14) * [9, 12]	(15) * [10, 13]

Both branches in this tree are closed. Hence, the tree is closed. It follows that the tableau constitutes a derivation of the conclusion from the premise in every system that includes T-OC. Since, these systems are sound with respect to the class of all frames that satisfies C-OC, the conclusion is a consequence of the premise in the class of all frames that satisfies this condition.

Systems of this kind rule out moral dilemmas of the following form: $OA \wedge OB \wedge \neg\Diamond(A \wedge B)$. $\neg((Op \wedge Oq) \wedge \neg\Diamond(p \wedge q))$ is a theorem. This seems to me to be a plausible view. (See Rönnedal (2012b, pp. 75–96) for more on moral dilemmas.)

Now, consider the following argument.

Argument 2

You ought to be completely honest.

It is necessary that if you are completely honest, then you do not lie.

Hence, it is forbidden that you lie.

Argument 2 is also intuitively valid; it seems necessary that if the premises are true then the conclusion is true too. We can show that the conclusion is derivable from the premises in every tableau system that includes the tableau rule T-MO. Here is a symbolization of argument 2: $Oh, \Box(h \supset \neg l) : Fl$, where h = You are completely honest, and l = You lie.

(1) $Oh, 0$
(2) $\Box(h \supset \neg l), 0$
(3) $\neg Fl, 0$
(4) $Pl, 0$ [3, ¬F]
(5) $0s1$ [4, P]
(6) $l, 1$ [4, P]
(7) $h, 1$ [1, 5, O]
(8) $0r1$ [5, T-MO]
(9) $h \supset \neg l, 1$ [2, 8, □]
↙ ↘

(10) $\neg h, 1$ [9, ⊃]	(11) $\neg l, 1$ [9, ⊃]
(12) * [7, 10]	(13) * [6, 11]

Both branches in this tree are closed. So, the tree itself is closed. This shows that the conclusion is derivable from the premises in every tableau system that includes T-MO. Since systems of this kind are sound with respect to the class of all frames that satisfies C-MO, the conclusion follows from the premises in all C-MO-frames.

This seems to be intuitively reasonable. It is a kind of means-end reasoning. In fact, $(OA \wedge \Box(A \supset B)) \supset OB$ is derivable in every system that includes T-MO. This is a version of the so-called, means-end principle that says that every necessary consequence of an obligation is obligatory.

We will now show how our systems can be used to establish that an argument is not valid. Consider the following argument.

Argument 3
You should give money to some charity.
It is necessary that if you give money to Oxfam, then you give money to some charity.
Hence, you ought to give money to Oxfam.

This argument is similar to argument 2, and it might seem to be valid. Doesn't it involve a kind of means-end reasoning that is plausible? However, on closer examination, we see that the second premise says that giving money to Oxfam is a *sufficient* condition for giving money to some charity, not a *necessary* means or consequence. There are many ways of giving money to some charity and perhaps some other way is better. Therefore, we cannot exclude the possibility that the premises are true while the conclusion is false. Of course, it might be true that you ought to give money to some charity and also true that you ought to give money to Oxfam, but this doesn't entail that the conclusion follows from the premises.

Argument 3 can be symbolized in our systems in the following way: Og, $\Box(o \supset g)$: Oo, where g = You give money to some charity, and o = You give money to Oxfam. We can show that this deduction isn't derivable in any of our systems and that the conclusion doesn't follow from the premises in any class of frames we have described. First we will show that the conclusion isn't derivable from the premises in the weakest system T.

(1) Og, 0
(2) $\Box(o \supset g)$, 0
(3) $\neg$Oo, 0
(4) P$\neg$o, 0 [3, $\neg$O]
(5) 0s1 [4, P]
(6) $\neg$o, 1 [4, P]
(7) g, 1 [1, 5, O]

At this stage the tableau is complete and open, i.e. we have applied every T-rule we can. We can use the open branch to read off a countermodel. W =

$\{w_0, w_1\}$, S = $\{<w_0, w_1>\}$, R is empty and g is true and o false in w_1. Since g is true in w_1 and w_1 is the only deontically accessible world from w_0, Og is true in w_0. $\Box(o \supset g)$ is vacuously true in w_0 since no possible world is alethically accessible from w_0. However, Oo is false in w_0. For o is false in w_1 and w_1 is deontically accessible from w_0. So, all premises are true in w_0, while the conclusion is false. Hence, this model shows that the argument isn't valid in the class of all alethic-deontic frames. However, it doesn't establish that the conclusion doesn't follow from the premises in some subset of this class. Nevertheless, we can show that the conclusion doesn't follow from the premises in any class of frames we describe in this essay. To do this we extend our countermodel with the following information: Sw_1w_1, Rw_0w_0, Rw_1w_1, Rw_0w_1, Rw_1w_0, o is false in w_0. It follows that the conclusion isn't derivable from the premises in any tableau system we consider in this paper.

These examples illustrate the usefulness of our alethic-deontic systems.

6. Deductively equivalent systems

We have mentioned two special alethic-deontic systems: Strong alethic-deontic logic and Full alethic-deontic logic. Full alethic-deontic logic is the system aTDB45dDT′B′45adMOOCMO′OC′45OMPPMP, and Strong alethic-deontic logic is the system aTDB45dT′B′45adMOMO′OC′45OMPPMP. So, FADL includes all tableau rules we have introduced in this essay except T-MOP, and StADL includes all tableau rules except T-dD, T-OC and T-MOP. For our purposes, FADL and StADL are especially interesting since they correspond to the class of all frames where the deontic accessibility relation is defined in terms of the alethic accessibility relation (according to Def(S)), and where the alethic accessibility relation is an equivalence relation. In the case of FADL, we also assume condition C-adD ($\forall x \exists y(Rxy \wedge My)$). There are many "weaker" systems, i.e. systems with fewer tableau rules, that are deductively equivalent, i.e. contain exactly the same theorems, with FADL or StADL. The following theorem mentions some of these.

Theorem 7. (i) The following systems are deductively equivalent with FADL: aB4dDadMO4, aB4dDadMO5, aB5dDadMO4, aB5dDadMO5, aT5dDadMO4, aT5dDadMO5, aB4dadMOOC4, aB4dadMOOC5, aB5dadMOOC4, aB5dadMOOC5, aT5dadMOOC4, and aT5dadMOOC5. (ii) The following systems are deductively equivalent with StADL: aTB4adMO4, and aTB4adMO5.

Proof. Left to the reader. The appendix in Rönnedal (2012b) may be useful.

7. Soundness and completeness theorems

Let $S = aA_1 \ldots A_n dB_1 \ldots B_n adC_1 \ldots C_n$ be a normal alethic-deontic tableau system, where $A_1 \ldots A_n$ is some subclass of our (non-basic) a-Rules, $B_1 \ldots B_n$ is some subclass of our (non-basic) d-Rules and $C_1 \ldots C_n$ is some subclass of our (non-basic) ad-Rules. Then we shall say that the class of frames, **F**, corresponds to S just in case **F** = **F**(C-A_1, …, C-A_n, C-B_1, …, C-B_n, C-C_1, …, C-C_n).

S is strongly sound with respect to **F** iff $\Gamma \vdash_S A$ entails $\Gamma \Vdash_{\mathbf{F}} A$. S is strongly complete with respect to **F** just in case $\Gamma \Vdash_{\mathbf{F}} A$ entails $\Gamma \vdash_S A$.

Theorem 8 (Soundness theorem). Let S be any of our normal alethic-deontic tableau systems and let **F** be the class of frames that corresponds to S. Then S is strongly sound with respect to **F**.

Proof. See Rönnedal (2012) and/or Rönnedal (2012b). ■

Theorem 9 (Completeness theorem). Let S be any of our normal alethic-deontic tableau systems and let **F** be the class of frames that corresponds to S. Then S is strongly complete with respect to **F**.

Proof. See Rönnedal (2012) and/or Rönnedal (2012b). ■

From the soundness and completeness theorems and theorems 1–3 it follows that Strong alethic-deontic logic is the system that is appropriate for $\mathbf{F}_s$(Eq) and that Full alethic-deontic logic is the system that is appropriate for $\mathbf{F}_s$(Eq, C-adD).

References

Anderson, A. R. (1956). The formal analysis of normative systems. In N. Rescher (ed.). *The Logic of Decision and Action*. Pittsburgh: University of Pittsburgh Press, 1967, pp. 147–213.

Anderson, A. R. (1958). A reduction of deontic logic to alethic modal logic. *Mind*, Vol. 67, No. 265, pp. 100–103.

Anderson, A. R. (1959). On the logic of commitment. *Philosophical Studies* 10, pp. 23–27.

Anderson, A. R. (1967). Some Nasty Problems in the Formal Logic of Ethics. *Noûs*, Vol. 1, No. 4, pp. 345–360.

Blackburn, P., de Rijke, M. & Venema, Y. (2001). *Modal Logic*. Cambridge University Press.

Blackburn, P., van Benthem, J. & Wolter, F. (eds.). (2007). *Handbook of Modal Logic*. Elsevier.

Chellas, B. F. (1980). *Modal Logic: An Introduction*. Cambridge: Cambridge University Press.

Feldman, F. (1986). *Doing The Best We Can: An Essay in Informal Deontic Logic*. Dordrecht: D. Reidel Publishing Company.

Fine, K. & Schurz, G. (1996). Transfer Theorems for Multimodal Logics. In J. Copeland (ed.). (1996). *Logic and Reality. Essays in Pure and Applied Logic. In Memory of Arthur Prior*. Oxford University Press, Oxford, pp. 169–213.

Fitting, M. & Mendelsohn, R. L. (1998). *First-Order Modal Logic*. Kluwer Academic Publishers.

Gabbay, D. M. (1976). *Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics*. Dordrecht: D. Reidel Publishing Company.

Gabbay, D., Horty, J., Parent, X., van der Meyden, E. & van der Torre, L. (eds.). (2013). *Handbook of Deontic Logic and Normative Systems*. College Publications.

Gabbay, D. M. & Guenthner, F. (eds.). (2001). *Handbook of Philosophical Logic 2nd Edition*, Vol. 3, Dordrecht: Kluwer Academic Publishers.

Gabbay, D. M., Kurucz, A., Wolter, F. & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*. Amsterdam: Elsevier.

Garson, J. W. (2006). *Modal Logic for Philosophers*. New York: Cambridge University Press.

Girle, R. (2000). *Modal Logics and Philosophy*. McGill-Queen's University Press.

Hilpinen, R. (ed.). (1971). *Deontic Logic: Introductory and Systematic Readings*. Dordrecht: D. Reidel Publishing Company.

Hilpinen, R. (ed.). (1981). *New Studies in Deontic Logic Norms, Actions, and the Foundation of Ethics*. Dordrecht: D. Reidel Publishing Company.

Kanger, S. (1957). New Foundations for Ethical Theory. Stockholm. Reprinted in *Hilpinen* (ed.) (1971), pp. 36–58.

Kracht, M. (1999). *Tools and Techniques in Modal Logic*. Amsterdam: Elsevier.

Kracht, M. & Wolter, F. (1991). Properties of Independently Axiomatizable Bimodal Logics. *The Journal of Symbolic Logic*, vol. 56, no. 4, pp. 1469–1485.

Lewis, C. I. & Langford, C. H. (1932). *Symbolic Logic*. New York: Dover Publications. Second edition 1959.

Popkorn, S. (1994). *First Steps in Modal Logic*. Cambridge University Press.

Rescher, N. (ed.). (1967). *The Logic of Decision and Action*. Pittsburgh: University of Pittsburgh Press.

Segerberg, K. (1971). *An Essay in Classical Modal Logic*. 3 vols. Uppsala: University of Uppsala.

Zeman, J. J. (1973). *Modal Logic: The Lewis-Modal Systems*. Oxford: Clarendon Press.

Rönnedal, D. (2010). *An Introduction to Deontic Logic*. Charleston, SC.

Rönnedal, D. (2012). Bimodal Logic. *Polish Journal of Philosophy*. Vol. VI, No. 2, pp. 71–93.

Rönnedal, D. (2012b). *Extensions of Deontic Logic: An Investigation into some Multi-Modal Systems*. Department of Philosophy, Stockholm University.

Rönnedal, D. (2012c). Temporal alethic-deontic logic and semantic tableaux. *Journal of Applied Logic*, 10, pp. 219–237.

Rönnedal, D. (2015). Alethic-Deontic Logic: Some Theorems. *Filosofiska Notiser*, Årgång 2, Nr. 1, pp. 61–77.

Rönnedal, D. (2015b). Alethic-Deontic Logic and the Alethic-Deontic Octagon. *Filosofiska Notiser*, Årgång 2, Nr. 3, pp. 27–68.

von Wright, G. H. (1971). Deontic Logic and the Theory of Conditions. In *Hilpinen* (ed.). (1971), pp. 159–177.

Åqvist, L. (1987). *Introduction to Deontic Logic and the Theory of Normative Systems*. Naples: Bibliopolis.

Åqvist, L. (2002). Deontic Logic. In Gabbay & Guenthner (eds.). *Handbook of Philosophical Logic 2nd Edition*. Vol. 8, Dordrecht: Kluwer Academic Publishers, pp. 147–264.

Daniel Rönnedal
Department of Philosophy
Stockholm University
daniel.ronnedal@philosophy.su.se