Input/output logics without weakening

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Abstract

Makinson and van der Torre introduced a number of input/output (I/O) logics to reason about conditional norms. The key idea is to make obligations relative to a given set of conditional norms. The meaning of the normative concepts is, then, given in terms of a set of procedures yielding outputs for inputs. Using the same methodology, Stolpe has developed some more I/O logics to include systems in which the rule of weakening of the output (or principle of inheritance) is replaced by a rule of closure under logical equivalence. We extend Stolpe's account in two directions. First, we show how to make it support reasoning by cases–a common form of reasoning. Second, we show how to inject a new (as we call it, aggregative) form of cumulative transitivity, which we think is more suitable for normative reasoning. The main outcomes of the paper are soundness and completeness theorems for the proposed systems with respect to their intended semantics.

1 Introduction

Deontic logic formalises reasoning with norms. It is relevant for a number of areas in computer science and artificial intelligence, like in particular normative multi-agent systems [13, 9] and machine ethics [2].

The present paper focuses on so-called input/output logics (I/O logics) as initially put forth by Makinson and van der Torre [26]. These aim at generalizing the theory of conditional obligation from modal and conditional logic [18, 24] to the abstract study of conditional codes viewed as relations between Boolean formulae. The meaning of the normative concepts is given in terms of a set of procedures yielding outputs for inputs. Detachment (or modus-ponens) is the core mechanism of the semantics being used. As argued by Boghossian [10], detachment is part of the meaning of a conditional statement.[1] The proof theory is formulated in terms of inference rules operating on pairs (a,x) of formulae (read *If a, then x is obligatory*) rather than on formulae.

The proposed framework can be based on either classical logic or intuitionistic logic [35, 32]. It can be extended in order to model different notions of permission [28, 8, 42], and in order to handle more complex phenomena like norm violation [27], reasoning about conflicting norms and about priorities [31, 46] and norm change [6, 7]. It has been applied to other areas of knowledge representation, like non-monotonic reasoning [27, 31], causal reasoning [4, 5] and argumentation theory [23]. Connections

[1] According to him, the disposition to reason according to modus-ponens is constitutive of the possession of the concept of conditional, and thus of the concept of norm. Note that such a motivation is not in the original papers [26, 27]. It is given and discussed in more detail in [34].

with other formalisms in knowledge representation and philosophical logic have also been found and studied [27, 44, 16, 43]. For an overview of input/output logics, the reader is referred to the handbook chapter on I/O logic [34]. A collection of benchmark examples can be found in [37]. The interested reader will also find in [3] an implementation in Isabelle/HOL of various deontic logics, including the original I/O logics for obligation.

These developments are not germane for our main purpose in this paper, and so we will leave them aside. Our primary contribution is to develop variants of the original input/output logics with the following two salient features:

- First, they do not satisfy the rule of "weakening the output" (WO). This is the rule: from (a, x) infer (a, y) whenever $x \vdash y$, where $\vdash$ is the deducibility relation used in classical propositional logic. This rule echoes the well-known rule of "right weakening" (RW) used in non-monotonic logic [21]: from $a \mid\!\sim x$ infer $a \mid\!\sim y$ whenever $x \vdash y$, where $\mid\!\sim$ stands for the non-monotonic inference relation. All the input/output logics of Makinson and van der Torre satisfy the rule WO. Those studied in this paper satisfy instead the rule "from (a, x) infer (a, y) whenever $x \vdash y$ and $y \vdash x$". In this respect, norms remain closed under logical equivalence.

- Second, instead of satisfying the traditional rule of "cumulative transitivity" (CT), they satisfy a variant rule called "aggregative cumulative transitivity" (ACT) discussed by van der Torre [45]. CT is also familiar from the literature on non-monotonic logic. This is the rule: from (a, x) and $(a \wedge x, y)$ infer (a, y). ACT is the rule: from (a, x) and $(a \wedge x, y)$ infer $(a, x \wedge y)$. It is the deontic analogue of the rule of "conjunctive cumulative transitivity" discussed by Verheij [47] in the context of the study of so-called abstract argumentative systems.

WO and CT are widely accepted for ontic conditionals. Their counterpart for deontic conditionals has generated some controversy.

First, it is known that WO creates a problem when it comes to handling normative dilemmas. There is a normative dilemma, when we both have (a, x) and $(a, \neg x)$. Goble [15] presents an in-depth discussion of all the issues surrounding the notion of normative dilemma. For present purposes we shall just mention the fact that the problem essentially arises from the interplay between WO and the so-called AND rule: from (a, x) and (a, y), infer $(a, x \wedge y)$. The following derivation shows that a system containing these two rules will not allow for deontic dilemmas without deontic "explosion":

$$\text{WO}\ \dfrac{\text{AND}\ \dfrac{(a,x) \qquad (a,\neg x)}{(a, x \wedge \neg x)}}{(a,y)}$$

There are two ways to prevent deontic explosion. One consists in letting AND go, and the other consists in letting WO go. In this paper, we follow the second approach, because there are independent reasons for taking WO out, to which we now turn.

A second lesser-known (but no less compelling) reason for letting WO go may be given in relation with norm compliance checking. WO yields the rule of "conjunction elimination" as a special case. This is the rule: from $(a,\ x \wedge y)$ infer $(a,\ x)$. Goble [14, p.183–184] and Hansen [17, §6.2], among others, have argued against conjunction elimination. The rule creates problems when assessing the level of compliance with the norms. There are cases where the two states of affairs (mentioned in the obligation) are only conjunctively required. If the obligation of x alone was derived, then when assessing how well or badly the agent did a strange consequence would follow, in the event that the agent made x, but not y, true. One would have to acknowledge that "he's not a complete scoundrel" [14, p.183], since at least one obligation (albeit a derived one) was fulfilled. Intuitively, one would like to be able to say that *no* obligations have been fulfilled, and that *nothing* right has happened. This may be illustrated with the following example.

Example 1 ("Sing and dance!" [14]) *Suppose there is a party of song and dance performers given in honour of Gene. Everyone ought to perform a song and dance routine, because Gene loves them both, and cannot tolerate either without the other. One guest, Fred, chooses not to sing but only to dance. Gene is appalled. The party is ruined, because of Gene's tantrum.*

We now briefly explain our main reason for using ACT in place of CT. It is the following one: counter-examples to CT may be found in the literature [22, 19, 25]; these are blocked, when ACT is used in place of CT. This is because they all rely on the intuition that the obligation of y ceases to hold when the obligation of (a,x) is violated. Due to Broome, the following example may be used to illustrate this point. We use the standard notation $(\top,x)$ for the unconditional obligation of x, where $\top$ is a zero-place connective standing for "tautology".

Example 2 (Marathon [12]) *Suppose you have entered the marathon. Consider the following instantiation of CT:*

You ought to exercise hard everyday	$(\top,x)$
If you exercise hard everyday, you ought to eat heartily	(x,y)
$\therefore$ *You ought to eat heartily*	$\therefore (\top,y)$

The conclusion seems to be counter-intuitive. Intuitively, the obligation to eat heartily no longer holds, if you take no exercise. In this example, the correct conclusion is $(\top,x \wedge y)$*, and not* $(\top,y)$*. Thus, ACT appears to be more suitable for normative reasoning, because it keeps track of what has been previously detached.*

Of course, given WO, ACT implies CT. This gives another independent reason for letting WO go−besides the aforementioned ones. In the presence of the latter rule, we would not get the chaining of rules right.

The layout of this paper is as follows. Section 2 lays the groundwork, and looks at the case of one-step detachment. More precisely, we consider the first two standard I/O operations defined by Makinson and van der Torre, called "simple-minded" (out_1) and "basic" (out_2), respectively. Both develop output by detachment. out_1 spells out the basic mechanism, and out_2 extends it in order to handle disjunctive inputs. For each of these I/O operations a variant is defined that does not satisfy WO. Section 3 goes one step further, and shows how to handle iteration of successive detachments. The focus is on the standard I/O operations called "reusable" (out_3) and "basic reusable" (out_4) by Makinson and van der Torre. We introduce a variant of each for which WO fails and for which rule chaining is supported in the form of ACT. In each case, the proposed variation is provided with equivalent axiomatic and semantic characterisations.

This paper is an extended version of an earlier paper presented at DEON 2014 (item [35] in the references). It builds on previous research by Stolpe [40, 41], who began work on versions of input/output logic without WO. His focus is on out_1 and out_3. In [36, 38] we study variant systems, in which in addition a consistency proviso restricts the application of AND and ACT. The study [30] revisits the notion of permission as described in [28] from the perspective of these new systems.

2 Single-step Detachment

2.1 Preliminaries

First, some definitions are needed. A normative code is a set N of conditional obligations. A conditional obligation is a pair (a, x), where a and x are two formulae of classical propositional logic. We use this notation instead of $\bigcirc(x \mid a)$, because the latter has distinct interpretations in the literature. In the notation (a, x), the first element a is called the body of the rule, and is thought of as an input, representing some condition or situation. The second element x is called the head of the rule, and is thought of as an output, representing what the norm tells us to be obligatory in that situation. In I/O logic, the main construct has the form

$$x \in out(N, a)$$

Intuitively: given input a (state of affairs), x (obligation) is in the output under norms N. An equivalent notation is: $(a, x) \in out(N)$. The I/O operations to be defined in this paper will be denoted by the symbol $\mathcal{O}$ in order to avoid any confusion with *out*.

Some further notation. $\mathcal{L}$ is the set of all formulae of classical propositional logic. Given an input $A \subseteq \mathcal{L}$, and a set N of norms, $N(A)$ denotes the image of N under A, i.e., $N(A) = \{x : (a, x) \in N \text{ for some } a \in A\}$. $Cn(A)$ denotes the set $\{x : A \vdash x\}$. The notation $x \dashv\vdash y$ is short for $x \vdash y$ and $y \vdash x$. Given $M \subseteq N$, we denote by $h(M)$ the set of all the heads of elements of M, viz $h(M) = \{x : (a, x) \in M\}$.

2.2 Simple-minded I/O Operation

We start with the simple-minded I/O operation out_1. The I/O operation to be defined here is noted $\mathcal{O}_1$. It is essentially a variation on the I/O operation PN_1 put forth by Stolpe [40, 41]. The main reason for including such an operation in our study is that the completeness result for it will be needed for subsequent developments.

Definition 3 (Semantics) *$x \in \mathcal{O}_1(N, A)$ if and only if there is some finite $M \subseteq N$ such that*

- *$M(Cn(A)) \neq \emptyset$, and*

- $x \dashv\vdash \bigwedge M(Cn(A))$

Intuitively: x is equivalent to the conjunction of heads of rules in some $M \subseteq N$ that are all triggered by input A.

The main difference between $\mathcal{O}_1$ and PN_1 arises when A does not trigger any norm, viz. $M(Cn(A)) = \varnothing$ for all $M \subseteq N$. In this limiting case, PN_1 outputs the set of all tautologies, while $\mathcal{O}_1$ outputs nothing. Von Wright [48, pp. 152–4] argues, rightly in our view, that the obligation of $\top$ does not express a genuine prescription.

$\mathcal{O}_1$ is monotonic with respect to the input set. The latter claim requires a careful and detailed proof, because there is a pitfall to avoid.

Theorem 4 (Factual monotony) *$\mathcal{O}_1(N,A) \subseteq \mathcal{O}_1(N,B)$ if $Cn(A) \subseteq Cn(B)$.*

Proof. Assume $x \in \mathcal{O}_1(N,A)$ and $Cn(A) \subseteq Cn(B)$. From the former, there is some finite $M_1 \subseteq N$ such that $M_1(Cn(A)) \neq \varnothing$, and

1. $x \dashv\vdash \bigwedge M_1(Cn(A))$

There is no guarantee that input set B does not trigger more pairs in M_1 than A does. To circumvent this problem, the argument takes a detour through the set

$$M_1^- = \{(c,y) \in M_1 : c \in Cn(A)\}$$

Thus, M_1^- is M_1 "stripped of" all the pairs that are not triggered by A. We have $M_1(Cn(A)) = M_1^-(Cn(A))$. We also have $M_1^-(Cn(A)) = M_1^-(Cn(B))$, viz.

$$\{y : (c,y) \in M_1^-, c \in Cn(A)\} = \{y : (c,y) \in M_1^-, c \in Cn(B)\}$$

The $\subseteq$-direction follows from the second opening assumption, $Cn(A) \subseteq Cn(B)$. The $\supseteq$-direction follows from the definition of M_1^-. The argument may, then, be continued thus:

2. $x \dashv\vdash \bigwedge M_1^-(Cn(A))$
3. $x \dashv\vdash \bigwedge M_1^-(Cn(B))$

Thus, $x \in \mathcal{O}_1(N,B)$ as required. ■

It immediately follows that $\mathcal{O}_1(N,A) \subseteq \mathcal{O}_1(N,B)$ whenever $A \subseteq B$.

We set $\mathcal{O}_1(N) = \{(A,x) : x \in \mathcal{O}_1(N,A)\}$.

The notion of derivation is defined as in standard I/O logic (see, e.g., Parent et al. [33]) except that $(\top,\top)$ is not allowed to appear in a derivation unless it is explicitly given in the set N of assumptions. We write $(a,x) \in \mathcal{D}_i(N)$ when there is a derivation of (a,x) from N. The subscript i is used to distinguish the systems. To say that, given a set R of rules, there is a derivation of (a,x) from N amounts to saying that there is a sequence α_1, ..., α_n of pairs of formulae such that $\alpha_n = (a,x)$ and, for all i such that $1 \leq i \leq n-1$, either $\alpha_i \in N$ or α_i is obtained from preceding elements(s) in the sequence using a rule in R. The elements in the sequence are all pairs of the form (b,y). Derivation steps done in the base logic are not part of it.

Definition 5 (Proof system) *$(a,x) \in \mathcal{D}_1(N)$ if and only if there is a derivation of (a,x) from N using the rules {SI, EQ, AND}:*

$$SI\ \frac{(a,x) \qquad b \vdash a}{(b,x)} \qquad\qquad EQ\ \frac{(a,x) \qquad x \dashv\vdash y}{(a,y)}$$

$$AND\ \frac{(a,x) \qquad (a,y)}{(a, x \wedge y)}$$

Where A is a set of formulae, $(A,x) \in \mathcal{D}_1(N)$ means that $(a,x) \in \mathcal{D}_1(N)$, for some conjunction a of formulae, all taken from a (finite) subset of A. $\mathcal{D}_1(N,A)$ is $\{x : (A,x) \in \mathcal{D}_1(N)\}$.

Theorem 6 *$\mathcal{O}_1$ validates the rules of $\mathcal{D}_1$ (for individual formulae a).*

Proof.

For SI. The argument is virtually the same as in the proof of Theorem 4.

For AND. Assume $x \in \mathcal{O}_1(N,a)$ and $y \in \mathcal{O}_1(N,a)$. Thus, there are $M_1, M_2 \subseteq N$ such that $M_1(Cn(a)) \neq \varnothing$, $M_2(Cn(a)) \neq \varnothing$ and

1. $x \dashv\vdash \bigwedge M_1(Cn(a))$
2. $y \dashv\vdash \bigwedge M_2(Cn(a))$

Hence

3. $x \wedge y \dashv\vdash \bigwedge M_1(Cn(a)) \wedge (\bigwedge M_2(Cn(a)))$

Put $M_3 = M_1 \cup M_2$. We have

$$M_3(Cn(a)) = M_1(Cn(a)) \cup M_2(Cn(a))$$

One, then, gets

4. $x \wedge y \dashv\vdash \bigwedge M_3(Cn(a))$

Thus, $x \wedge y \in \mathcal{O}_1(N,a)$ as required.

For EQ, the argument is straightforward, and is omitted. (Remember that, if $a \dashv\vdash b$, then $Cn(a) = Cn(b)$.) ∎

Soundness tells us that what is derivable is also valid; that is, if (according to the proof theory) the pair (A,x) is derivable from N, then (according to the semantics) given input A x is in the output under norms N. Formally, this can be written as shown below. Like in modal logic, we distinguish between a weak and a strong version of the theorem.

Theorem 7 (Soundness, weak version) $\mathcal{D}_1(N,a) \subseteq \mathcal{O}_1(N,a)$.

Proof. Assume $x \in \mathcal{D}_1(N,a)$, viz $(a,x) \in \mathcal{D}_1(N)$. Let α_1, ..., α_n be a derivation of (a,x). We show by induction on i that, for all $1 \leq i \leq n$, $\alpha_i \in \mathcal{O}_1(N)$.

For the basis of the induction, where α_1 is, e.g., (a,x), the argument is straightforward. By the definition of the notion of derivation, $(a,x) \in N$. Put $M = \{(a,x)\}$. We have $M(Cn(a)) = \{x\}$, and so $x \in \mathcal{O}_1(N,a)$ by Definition 3, and then $(a,x) \in \mathcal{O}_1(N)$.

For the inductive part of the proof, we assume as an inductive hypothesis that $\alpha_j \in \mathcal{O}_1(N)$ for all $j < i$, and argue that $\alpha_i \in \mathcal{O}_1(N)$ too. Either i) $\alpha_i \in N$ or ii) α_i is obtained from previous pairs using a rule. In case i), a similar argument as for the base case yields $\alpha_i \in \mathcal{O}_1(N)$. In case ii), the proof goes as follows.

- $\alpha_i = (a_i, x_i)$ is obtained using SI. In this case, there is some $j < i$ such that $\alpha_j = (a_j, x_i) \in \mathcal{D}_1(N)$ and $a_j \vdash a_i$. By the inductive hypothesis, $(a_j, x_i) \in \mathcal{O}_1(N)$, i.e. $x_i \in \mathcal{O}_1(N, a_j)$. By Theorem 6, $x_i \in \mathcal{O}_1(N, a_i)$, and so $\alpha_i \in \mathcal{O}_1(N)$ as required.

- $\alpha_i = (a_i, x_i)$ is obtained using AND. In this case, x_i is of the form $x_j \wedge x_k$, with $k, j < i$ and $\alpha_j = (a_i, x_j) \in \mathcal{D}_1(N)$ and $\alpha_k = (a_i, x_k) \in \mathcal{D}_1(N)$. By the inductive hypothesis, $(a_i, x_j) \in \mathcal{O}_1(N)$ and $(a_i, x_k) \in \mathcal{O}_1(N)$, viz $x_j \in \mathcal{O}_1(N, a_i)$ and $x_k \in \mathcal{O}_1(N, a_i)$. By Theorem 6, $x_j \wedge x_k \in \mathcal{O}_1(N, a_i)$, and so $\alpha_i \in \mathcal{O}_1(N)$ as required.

- $\alpha_i = (a_i, x_i)$ is obtained using EQ. In this case, there is $j < i$ such that $\alpha_j = (a_i, x_j) \in \mathcal{D}_1(N)$ and $x_i \dashv\vdash x_j$. By the inductive hypothesis, $(a_i, x_j) \in \mathcal{O}_1(N)$, i.e. $x_j \in \mathcal{O}_1(N, a_i)$. By Theorem 6, $x_i \in \mathcal{O}_1(N, a_i)$, and so $\alpha_i \in \mathcal{O}_1(N)$ as required.

This ends the proof. ■

Theorem 8 (Soundness, strong version) $\mathcal{D}_1(N, A) \subseteq \mathcal{O}_1(N, A)$.

Proof. Let $x \in \mathcal{D}_1(N, A)$, i.e. $(A, x) \in \mathcal{D}_1(N)$. So there is a conjunction a of elements of A such that $(a, x) \in \mathcal{D}_1(N)$. By Theorem 7, $(a, x) \in \mathcal{O}_1(N)$, and thus $x \in \mathcal{O}_1(N, a)$. But $Cn(a) \subseteq Cn(A)$. By Theorem 4, $x \in \mathcal{O}_1(N, A)$. ■

Completeness tells us that what is valid is also derivable; that is, if (according to the semantics) given input A x is in the output under norms N, then (according to the proof theory) the pair (A, x) is derivable from N. Formally, this can be written as:

Theorem 9 (Completeness, strong version) $\mathcal{O}_1(N, A) \subseteq \mathcal{D}_1(N, A)$.

Proof. Assume $x \in \mathcal{O}_1(N, A)$. So there exists some finite $M \subseteq N$ such that $M(Cn(A)) = \{x_1, ..., x_n\} \neq \emptyset$ and $x \dashv\vdash \bigwedge_{i=1}^{n} x_i$. For each x_i, there is some $a_i \in Cn(A)$ such that $(a_i, x_i) \in M$. For each a_i, there is also a conjunction b_i of elements in A such that $b_i \vdash a_i$. A derivation of (A, x) from M, and hence from N, is shown below.

$$\dfrac{\dfrac{\dfrac{(a_1, x_1)}{(\bigwedge_{i=1}^{n} b_i, x_1)}\,\text{SI} \quad \cdots\cdots \quad \dfrac{(a_n, x_n)}{(\bigwedge_{i=1}^{n} b_i, x_n)}\,\text{SI}}{(\bigwedge_{i=1}^{n} b_i, \bigwedge_{i=1}^{n} x_i)}\,\text{AND}}{(\bigwedge_{i=1}^{n} b_i, x)}\,\text{EQ}$$

This is a derivation of (A, x), as $\bigwedge_{i=1}^{n} b_i$ is a conjunction of elements in A. Hence $x \in \mathcal{D}_1(N, A)$ as required. ■

2.3 Basic I/O Operation

In this section, the account described in the previous section is extended to the basic I/O operation out_2, which supports reasoning by cases, viz. the rule

$$\text{OR}\quad \frac{(a,x) \qquad (b,x)}{(a \vee b, x)}$$

The I/O operation is denoted $\mathcal{O}_2$, and the corresponding proof system is called $\mathcal{D}_2$.

A set V of wffs is said to be maximal consistent whenever it is consistent and none of its proper extensions is consistent. That is: $V \nvdash \bot$; and if $y \notin V$, then $V \cup \{y\} \vdash \bot$. V is said to be complete if it is either equal to $\mathcal{L}$ or maximal consistent.

Definition 10 (Semantics) $\mathcal{O}_2(N,A) = \cap\{\mathcal{O}_1(N,V) : A \subseteq V, V$ *complete*$\}$.

Theorem 11 $\mathcal{O}_1(N,A) \subseteq \mathcal{O}_2(N,A)$.

Proof. Let $x \in \mathcal{O}_1(N,A)$. Let V be a complete set such that $A \subseteq V$. By Theorem 4, $x \in \mathcal{O}_1(N,V)$. By Definition 10, $x \in \mathcal{O}_2(N,A)$ as required. ■

Theorem 12 (Factual monotony) $\mathcal{O}_2(N,A) \subseteq \mathcal{O}_2(N,B)$ *if* $Cn(A) \subseteq Cn(B)$.

Proof. Assume $x \in \mathcal{O}_2(N,A)$ and $Cn(A) \subseteq Cn(B)$. Let V be a complete set such that $B \subseteq V$. We have $Cn(B) \subseteq Cn(V) = V$. From this and the second opening assumption, $Cn(A) \subseteq V$. So, $A \subseteq V$. From this and the first opening assumption, $x \in \mathcal{O}_1(N,V)$. Thus, $x \in \mathcal{O}_2(N,B)$. ■

Definition 13 (Proof theory) $(a,x) \in \mathcal{D}_2(N)$ *if and only if there is a derivation of* (a,x) *from N using the rules of* $\mathcal{D}_1$ *supplemented with*

$$OR\quad \frac{(a,x) \qquad (b,x)}{(a \vee b, x)}$$

The next theorem appeals to the fact that $\mathcal{O}_1$ validates AND and EQ for an input set of arbitrary cardinality rather than just a singleton set. The argument is virtually the same in both cases. Details are omitted.

Theorem 14 $\mathcal{O}_2$ *validates the rules of* $\mathcal{D}_2$ *(for individual formulae a).*

Proof. For SI. Assume $x \in \mathcal{O}_2(N,a)$ with $b \vdash a$. Let V be a complete set such that $b \in V$. From $b \vdash a$, we get $a \in V$. By Definition 10, we infer $x \in \mathcal{O}_1(N,V)$. This shows that $x \in \mathcal{O}_2(N,b)$.

For AND. Assume $x \in \mathcal{O}_2(N,a)$ and $y \in \mathcal{O}_2(N,a)$. Let V be a complete set such that $a \in V$. By Definition 10, $x \in \mathcal{O}_1(N,V)$ and $y \in \mathcal{O}_1(N,V)$. Since $\mathcal{O}_1$ validates AND, $x \wedge y \in \mathcal{O}_1(N,V)$. This shows that $x \wedge y \in \mathcal{O}_2(N,a)$.

For OR. Assume $x \in \mathcal{O}_2(N,a)$ and $x \in \mathcal{O}_2(N,b)$. Let V be a complete set containing $a \vee b$. Since V is complete, either $a \in V$ or $b \in V$. Assume that the first applies. In that case, $x \in \mathcal{O}_1(N,V)$, by the first opening assumption and Definition 10. Assume the second applies. In that case $x \in \mathcal{O}_1(N,V)$, by the second opening assumption and Definition 10. Either way, $x \in \mathcal{O}_1(N,V)$, and thus $x \in \mathcal{O}_2(N,a \vee b)$ as required.

For EQ, assume $x \in \mathcal{O}_2(N,a)$ and $x \dashv\vdash y$. Let V be a complete set containing a. By Definition 10, $x \in \mathcal{O}_1(N,V)$. Since $\mathcal{O}_1$ validates EQ, $y \in \mathcal{O}_1(N,V)$, and so $y \in \mathcal{O}_2(N,a)$ as required. ■

Theorem 15 (Soundness, strong version) $\mathcal{D}_2(N,A) \subseteq \mathcal{O}_2(N,A)$.

Proof. The proof is virtually the same as that for Theorem 8, but using Theorems 12 and 14. ■

Theorem 16 (Completeness, strong version) $\mathcal{O}_2(N,A) \subseteq \mathcal{D}_2(N,A)$.

Proof. We give an outline of the proof for a singleton input set $\{a\}$. The proof may easily be generalized to an input set of arbitrary cardinality. For ease of exposition, throughout the proof we write (SI,AND) to indicate an application of SI followed by that of AND. We break the argument into two cases.

Case 1: a is inconsistent. In this case, there is exactly one complete set V containing a; it is $\mathcal{L}$. So $\mathcal{O}_2(N,a) = \mathcal{O}_1(N,\mathcal{L})$. Let $x \in \mathcal{O}_1(N,\mathcal{L})$. This means that $x \dashv\vdash \bigwedge_{i=1}^{n} x_i$, for $x_1,...,x_n \in h(N)$. Let $a_1,...,a_n$ be the body of the rules in question. We have $a \vdash \bigwedge_{i=1}^{n} a_i$. A derivation of (a,x) from N may, then, be obtained as shown below.

$$\dfrac{\text{SI}\;\dfrac{\dfrac{(a_1,x_1) \quad \cdots \quad (a_n,x_n)}{(\bigwedge_{i=1}^{n} a_i, \bigwedge_{i=1}^{n} x_i)}\,(\text{SI,AND}) \qquad \bigwedge_{i=1}^{n} x_i \dashv\vdash x}{(\bigwedge_{i=1}^{n} a_i, x)}\,\text{EQ} \qquad a \vdash \bigwedge_{i=1}^{n} a_i}{(a,x)}$$

Case 2: a is consistent. Assume (for reductio) that $x \in \mathcal{O}_2(N,a)$ and that $x \notin \mathcal{D}_2(N,a)$. From the former, $x \dashv\vdash \bigwedge_{i=1}^{n} x_i$, for $x_1,...,x_n \in h(N)$. In order to derive the contradiction that $x \notin \mathcal{O}_2(N,a)$, we start by showing that $\{a\}$ can be extended to some "maximal" $V \supseteq \{a\}$ such that $x \notin \mathcal{D}_2(N,V)$. By maximal, we mean that for all $V' \supset V$, $x \in \mathcal{D}_2(N,V')$. Thus, V is amongst the "biggest" input sets V containing a and not making x derivable.

V is built from a sequence of sets $V_0,V_1,V_2,...$ as follows. Consider an enumeration $x_1,x_2,x_3,...$ of all the formulae. We define:

$$V_0 = \{a\}$$
$$V_n = \begin{cases} V_{n-1} \cup \{x_n\}, & \text{if } x \notin \mathcal{D}_2(N, V_{n-1} \cup \{x_n\}) \\ V_{n-1}, & \text{otherwise} \end{cases}$$
$$V = \cup\{V_n : n \geq 0\}$$

It is a straightforward matter to show the following:

Fact 1 $x \notin \mathcal{D}_2(N,V_n)$, *for all* $n \geq 0$.

Fact 2 $V_n \subseteq V$, *for all* $n \geq 0$.

Fact 3 *For every finite subset* $V' \subseteq V$, $V' \subseteq V_n$, *for some* $n \geq 0$.

By Fact 2, V includes $\{a\}$ (= V_0).The argument may be continued thus:

Claim 1 $x \notin \mathcal{D}_2(N, V)$.

Proof of the claim. Assume, to reach a contradiction, that $x \in \mathcal{D}_2(N, V)$. By compactness for $\mathcal{D}_2$, $x \in \mathcal{D}_2(N, V')$ for some finite $V' \subseteq V$. By Fact 3, $V' \subseteq V_n$ for some $n \geq 0$. By monotony in the right argument, $x \in \mathcal{D}_2(N, V_n)$. This contradicts Fact 1.

Claim 2 *For all* $V' \supset V$, $x \in \mathcal{D}_2(N, V')$.

Proof of the claim. Let $V' \supset V$. So, there is some y such that $y \in V'$ but $y \notin V$. Any such y is such that $y = x_n$, for some $n \geq 1$. By Fact 2, $V_n \subseteq V$. So, $y \notin V_n$. By construction, $V_{n-1} = V_n$, and $x \in \mathcal{D}_2(N, V_{n-1} \cup \{y\}) = \mathcal{D}_2(V, V_n \cup \{y\})$. But $V_n \cup \{y\} \subseteq V \cup \{y\} \subseteq V'$. By monotony in the right argument for $\mathcal{D}_2$, we get that $x \in \mathcal{D}_2(N, V')$, as required.

Claim 3 *V is consistent.*

Proof of the claim. Assume not. Since $x \dashv\vdash \bigwedge_{i=1}^{n} x_i$, for $x_1, ..., x_n \in h(N)$, a derivation of (V, x) from N may be obtained by reiterating the argument under case 1, contradicting Claim 1.

Claim 4 *V is $\neg$-complete; that is, for all y, either $y \in V$ or $\neg y \in V$.*

Proof of the claim. Assume $y \notin V$ and $\neg y \notin V$ for some y. By Claim 2, it follows that $x \in \mathcal{D}_2(N, V \cup \{y\})$ and $x \in \mathcal{D}_2(N, V \cup \{\neg y\})$. Thus, $(b \wedge y, x)$ and $(c \wedge \neg y, x)$ are both derivable from N, where b and c are conjunctions of elements of V. The following is, then, derivable:

$$\dfrac{\dfrac{(b \wedge y, x) \qquad (c \wedge \neg y, x)}{((b \wedge y) \vee (c \wedge \neg y), x)}\text{OR}}{(b \wedge c, x)}\text{SI}$$

Thus, $x \in \mathcal{D}_2(N, V)$, in contradiction with Claim 1.

Claim 5 *V is maximal consistent; that is, if $V \cup \{y\}$ is consistent, then $y \in V$.*

Proof of the claim. Assume $y \notin V$. By Claim 4, $\neg y \in V$. It, then, follows that $V \cup \{y\}$ is inconsistent, as required.

We are almost finished. By Theorem 8 and Theorem 9, we have $\mathcal{O}_1(N, V) = \mathcal{D}_1(N, V) \subseteq \mathcal{D}_2(N, V)$. So $x \notin \mathcal{O}_1(N, V)$. Hence, $x \notin \mathcal{O}_2(N, A)$. ∎

3 Iterated Case

3.1 Reusable I/O Operation

This section focuses on the question of how to handle iterations of successive detachments. We redefine Makinson and van der Torre's reusable output operation out_3 so that it validates neither WO nor CT but ACT:

$$\text{ACT}\ \frac{(a,x) \qquad (a\wedge x,y)}{(a,x\wedge y)} \qquad\qquad \text{CT}\ \frac{(a,x) \qquad (a\wedge x,y)}{(a,y)}$$

ACT and WO together imply CT.

Stolpe [40, 41] names "PN_3" his own variant of out_3. He writes that the distinctive rule of PN_3 is the rule of "mediated cumulative transitivity" (MCT):

$$\text{MCT}\ \frac{(a,x') \qquad x'\vdash x \qquad (a\wedge x,y)}{(a,y)}$$

In fact, given the other rules in Stolpe's system, MCT is equivalent to CT. This is easily checked. The other rules are: SI, AND and EQ. On the one hand, given reflexivity for $\vdash$, MCT entails CT. For assume (a,x) and $(a\wedge x,y)$. Since $x\vdash x$, a direct application of MCT yields (a,y). On the other hand, given SI, CT entails MCT:

$$\text{CT}\ \frac{(a,x') \qquad \dfrac{(a\wedge x,y) \qquad \dfrac{x'\vdash x}{a\wedge x'\vdash a\wedge x}}{(a\wedge x',y)}\ \text{SI}}{(a,y)}$$

Note that, given SI, ACT implies the aggregative version of MCT:

$$\text{AMCT}\ \frac{(a,x') \qquad x'\vdash x \qquad (a\wedge x,y)}{(a,x'\wedge y)}$$

In this respect, weakening has still a "ghostly" role to play for iteration of successive detachments.

For the sake of conciseness, throughout this section $\mathcal{B}_A^M$ will denote the set of all the Bs such that $A\subseteq B=Cn(B)\supseteq M(B)$. Intuitively, $\mathcal{B}_A^M$ gathers all the Bs that contain A and are closed under both Cn and M.

Definition 17 (Semantics) *$x\in\mathcal{O}_3(N,A)$ if and only if there is some finite $M\subseteq N$ such that,*

- *$M(Cn(A))\neq\emptyset$, and*
- *for all B, if $B\in\mathcal{B}_A^M$, then $x\dashv\vdash\bigwedge M(B)$.*

We do not single out any particular B as "proper". But we highlight two very useful such Bs, which we call the smallest and the largest: $\cap\mathcal{B}_A^M$; $\mathcal{L}$.

A subset M of N that makes $x\in\mathcal{O}_3(N,A)$ true is called an "A-witness for x". Unlike with $\mathcal{O}_1$, we have the guarantee that such a M does not contain any rule that is superfluous, viz. not required to get output x:

Theorem 18 *If M is an A-witness for x, then $x\dashv\vdash\bigwedge h(M)$.*

Proof. Let M be an A-witness for x. By Definition 17, $M(Cn(A))\neq\emptyset$, and $x\dashv\vdash\bigwedge M(B)$ for all $B\in\mathcal{B}_A^M$. Consider $B=\mathcal{L}$. We have $x\dashv\vdash\bigwedge M(\mathcal{L})$. But $M(\mathcal{L})=h(M)$, and thus $x\dashv\vdash\bigwedge h(M)$. ■

Theorem 19 (Factual monotony) *We have* $\mathcal{O}_3(N, A_1) \subseteq \mathcal{O}_3(N, A_2)$ *whenever* $Cn(A_1) \subseteq Cn(A_2)$.

Proof. Assume $x \in \mathcal{O}_3(N, A_1)$ and $Cn(A_1) \subseteq Cn(A_2)$. From the first, we get: there is some finite $M_1 \subseteq N$ such that $M_1(Cn(A_1)) \neq \emptyset$ and, for all $B \in \mathcal{B}^{M_1}_{A_1}$,

$$M_1(B) = \{x_1, ..., x_n\} \text{ and } x \dashv\vdash \bigwedge_{i=1}^{n} x_i \tag{1}$$

Note that, by Theorem 18, $x \dashv\vdash \bigwedge h(M_1)$, and so the detour made in the proof of Theorem 4 is no longer needed.

From $Cn(A_1) \subseteq Cn(A_2)$, we get $M_1(Cn(A_1)) \subseteq M_1(Cn(A_2))$. This implies that $M_1(Cn(A_2)) \neq \emptyset$. Now, consider some $B_1 \in \mathcal{B}^{M_1}_{A_2}$. We have $A_2 \subseteq B_1$. Therefore, $Cn(A_2) \subseteq Cn(B_1) = B_1$. From $A_1 \subseteq Cn(A_1) \subseteq Cn(A_2)$, we then get $A_1 \subseteq B_1$, and hence $B_1 \in \mathcal{B}^{M_1}_{A_1}$. By (1), $x \dashv\vdash \bigwedge M_1(B_1) \dashv\vdash \bigwedge h(M_1)$. So, $x \in \mathcal{O}_3(N, A_2)$ as required. ■

We define $\mathcal{O}_3(N) = \{(A, x) : x \in \mathcal{O}_3(N, A)\}$.

Example 20 shows that $\mathcal{O}_3$ does not validate the rule of deontic detachment, and hence does not validate CT.

Example 20 (Deontic detachment) *Let* $N = \{(\top, a), (a, x)\}$. *We have that* $a \in \mathcal{O}_3(N, \top)$, *since* $M = \{(\top, a)\}$ *is a* $\top$*-witness for a. We also have that* $x \in \mathcal{O}_3(N, a)$, *since* $M = \{(a, x)\}$ *is an a-witness for x. But we do not have* $x \in \mathcal{O}_3(N, \top)$. *This may be verified in two steps. First, we identify all the non-empty subsets M of N that are triggered by the input, in the sense that* $M(Cn(\emptyset)) \neq \emptyset$. *Next, we go through the list of all these subsets, and check that, for none of them, the smallest relevant B outputs heads whose conjunction is equivalent to x:*

M	B	$M(B)$
$\{(\top, a)\}$	$Cn(a)$	$\{a\}$
$\{(\top, a)(a, x)\}$	$Cn(a, x)$	$\{a, x\}$

Definition 21 (Proof system) $(a, x) \in \mathcal{D}_3(N)$ *if and only if there is a derivation of* (a, x) *from N using the rules {SI, EQ, ACT}.*

$$ACT\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)}$$

AND is derivable from SI and ACT. We define $(A, x) \in \mathcal{D}_3(N)$ and $\mathcal{D}_3(N, A)$ as we did for $\mathcal{D}_1$.

Theorem 22 $\mathcal{O}_3$ *validates the rules of* $\mathcal{D}_3$ *(for individual formulae a).*

Proof. The argument for SI is virtually the same as in the proof of Theorem 19. The argument for EQ is straightforward, and is omitted. We show ACT. Assume that $x \in$

$\mathcal{O}_3(N, a)$, $y \in \mathcal{O}_3(N, a \wedge x)$ and $x \wedge y \notin \mathcal{O}_3(N, a)$. From the first two, it follows that there are finite $M_1, M_2 \subseteq N$ such that $M_1(Cn(a)) \neq \varnothing$, $M_2(Cn(a, x)) \neq \varnothing$, and

$$x \dashv\vdash \bigwedge M_1(B) \text{ for all } B \in \mathcal{B}_a^{M_1} \tag{2}$$

$$y \dashv\vdash \bigwedge M_2(B) \text{ for all } B \in \mathcal{B}_{a \wedge x}^{M_2} \tag{3}$$

By Theorem 18,

$$x \dashv\vdash \bigwedge h(M_1) \tag{4}$$

$$y \dashv\vdash \bigwedge h(M_2) \tag{5}$$

Therefore,

$$x \wedge y \dashv\vdash \bigwedge h(M_1) \wedge (\bigwedge h(M_2)) \tag{6}$$

$$\dashv\vdash \bigwedge h(M_3) \tag{7}$$

where $M_3 = M_1 \cup M_2$. From the third opening assumption, since $M_3(Cn(a)) \neq \varnothing$, it follows that there is some $B_1 \in \mathcal{B}_a^{M_3}$ such that

$$\text{not-}(\ x \wedge y \dashv\vdash \bigwedge M_3(B_1)\ ) \tag{8}$$

We have $M_1(B_1) \subseteq M_3(B_1)$, and so $B_1 \in \mathcal{B}_a^{M_1}$. Therefore $x \in B_1$, and hence $a \wedge x \in B_1$. So $B_1 \in \mathcal{B}_{a \wedge x}^{M_2}$ too, since $M_2(B_1) \subseteq M_3(B_1)$. Now,

$$M_3(B_1) = M_1(B_1) \cup M_2(B_1)$$

where $\bigwedge M_1(B_1) \dashv\vdash x$ and $\bigwedge M_2(B_1) \dashv\vdash y$. Thus, $\bigwedge M_3(B_1) \dashv\vdash x \wedge y$. This contradicts equation (8) above. ■

Theorem 23 (Soundness, strong version) $\mathcal{D}_3(N, A) \subseteq \mathcal{O}_3(N, A)$.

Proof. The proof is virtually the same as that for Theorem 8 using Theorems 19 and 22. ■

For future reference, we note the following:

Remark 24 *ACT holds in the following variant forms*

$$x \in \mathcal{O}_3(N, A), y \in \mathcal{O}_3(N, A \cup \{x\}) \Rightarrow x \wedge y \in \mathcal{O}_3(N, A) \tag{ACT$_1$}$$

$$x \in \mathcal{O}_3(N, A), y \in \mathcal{O}_3(N, Cn(A \cup \{x\})) \Rightarrow x \wedge y \in \mathcal{O}_3(N, A) \tag{ACT$_2$}$$

Proof. For ACT$_1$, this is just a matter of rerunning the argument for ACT in the proof of Theorem 22, replacing everywhere a with A, and $a \wedge x$ and $\{a, x\}$ with $A \cup \{x\}$. ACT$_2$ follows from ACT$_1$ and factual monotony for $\mathcal{O}_3$, Theorem 19. Trivially, $CnCn(A \cup \{x\}) \subseteq Cn(A \cup \{x\})$, and so $\mathcal{O}_3(N, Cn(A \cup \{x\})) \subseteq \mathcal{O}_3(N, A \cup \{x\})$. ■

Theorem 25 (Completeness, strong version) $\mathcal{O}_3(N, A) \subseteq \mathcal{D}_3(N, A)$.

Proof. We give an outline of the proof for the particular case where A is a singleton set $\{a\}$. Suppose that $x \in \mathcal{O}_3(N,a)$. To show: $x \in \mathcal{D}_3(N,a)$. From the former, there is some finite $M \subseteq N$ such that $M(Cn(a)) \neq \emptyset$ and, for all $B \in \mathcal{B}_a^M$, $x \dashv\vdash \bigwedge M(B)$.

Put $B_1 = Cn(\{a\} \cup \mathcal{D}_3(M,a))$. We have $a \in B_1 = Cn(B_1)$. We also have $M(B_1) \neq \emptyset$, because $Cn(a) \subseteq B_1$. A phasing result established by Makinson and van der Torre [26] tells us that in the proof system corresponding to out_3 the rule WO can always be applied last. This allows, then, to establish that $M(B_1) \subseteq B_1$, so that $B_1 \in \mathcal{B}_a^M$. The opening assumption, then, yields, $x \dashv\vdash \bigwedge M(B_1)$.

Based on this, one gets a derivation of (a,x) from N as follows. First, note that $M(B_1) \neq \emptyset$. By Definition 3, one gets $x \in \mathcal{O}_1(N, \{a\} \cup \mathcal{D}_3(M,a))$. By Theorem 9, $x \in \mathcal{D}_1(N, \{a\} \cup \mathcal{D}_3(M,a))$, and thus $x \in \mathcal{D}_3(N, \{a\} \cup \mathcal{D}_3(M,a))$. This means that $x \in \mathcal{D}_3(N, \{a, a_1, ..., a_n\})$, where, for each a_i, $a_i \in \mathcal{D}_3(M,a)$. By AND, $\bigwedge_{i=1}^n a_i \in \mathcal{D}_3(M,a)$. Since $M \subseteq N$, $\bigwedge_{i=1}^n a_i \in \mathcal{D}_3(N,a)$. A derivation of (a,x) from N is shown below.

$$\dfrac{\text{ACT}\,\dfrac{(a, \bigwedge_{i=1}^n a_i) \qquad (a \wedge (\bigwedge_{i=1}^n a_i), x)}{(a, \bigwedge_{i=1}^n a_i \wedge x)} \qquad \dfrac{x \vdash \bigwedge_{i=1}^n a_i}{\bigwedge_{i=1}^n a_i \wedge x \dashv\vdash x}}{(a,x)}\,\text{EQ}$$

The argument for $x \vdash \bigwedge_{i=1}^n a_i$ appeals to two lemmas:

- $x \dashv\vdash \bigwedge h(M)$, Theorem 18
- $h(M) \vdash a_i$, for all $1 \leq i \leq n$ – the proof of this is by induction on the length of the derivation of (a, a_i)

The argument may be generalized to an input set A of arbitrary cardinality. ■

3.2 Basic Reusable I/O Operation

This section shows that the two accounts discussed in the previous sections may be combined to yield a new basic reusable operation out_4, with ACT and OR, but neither CT nor WO amongst its primitive rules. Our proposed treatment of $\mathcal{O}_4$ is similar to that of $\mathcal{O}_2$. We set:

Definition 26 (Semantics) $\mathcal{O}_4(N,A) = \cap\{\mathcal{O}_3(N,V) : A \subseteq V, V\ \textit{complete}\}$.

Theorem 27 $\mathcal{O}_3(N,A) \subseteq \mathcal{O}_4(N,A)$.

Proof. Let $x \in \mathcal{O}_3(N,A)$. Let V be a complete set such that $A \subseteq V$. By monotony for Cn, $Cn(A) \subseteq Cn(V)$. By Theorem 19, $x \in \mathcal{O}_3(N,V)$. By Definition 26, $x \in \mathcal{O}_4(N,A)$ as required. ■

Theorem 28 (Factual monotony) $\mathcal{O}_4(N,A) \subseteq \mathcal{O}_4(N,B)$ *if* $Cn(A) \subseteq Cn(B)$.

Proof. Assume $Cn(A) \subseteq Cn(B)$. Let $x \in \mathcal{O}_4(N,A)$. To show: $x \in \mathcal{O}_4(N,B)$. Let V be a complete set such that $B \subseteq V$. By monotony for Cn, $Cn(B) \subseteq Cn(V)$. Also $Cn(V) = V$. Hence $Cn(B) \subseteq V$. From this and the opening assumption, $Cn(A) \subseteq V$. So, $A \subseteq V$. From this and $x \in \mathcal{O}_4(N,A)$, it follows that $x \in \mathcal{O}_3(N,V)$ by Definition 26. Thus, $x \in \mathcal{O}_4(N,B)$ as required. ■

Deontic detachment still fails, as illustrated below.

Example 29 (Deontic detachment) *Put* $N = \{(\top, a), (a, x)\}$. *In Example 20, on p. 200, we pointed out that* $a \in \mathcal{O}_3(N, \top)$ *and* $x \in \mathcal{O}_3(N, a)$. *By Theorem 27,* $a \in \mathcal{O}_4(N, \top)$ *and* $x \in \mathcal{O}_4(N, a)$. *But* $\mathcal{L}$ *qualifies as a complete set extending* $\{\top\}$. *We have:*

M	B	$M(B)$
$\{(\top, a)\}$	$\mathcal{L}$	$\{a\}$
$\{(\top, a), (a, x)\}$	$\mathcal{L}$	$\{a, x\}$

Thus, $x \notin \mathcal{O}_3(N, \mathcal{L})$, *and hence* $x \notin \mathcal{O}_4(N, \top)$.

Definition 30 (Proof theory) $(a, x) \in \mathcal{D}_4(N)$ *if and only if there is a derivation of* (a, x) *from N using the rules of* $\mathcal{D}_3$ *supplemented with*

$$OR \quad \frac{(a, x) \qquad (b, x)}{(a \vee b, x)}$$

The following lemma will come in handy:

Lemma 31 *If V is a complete set, then* $Cn(V \cup \{x\})$ *is also a complete set.*

Proof. Either i) $x \in V$ or ii) $x \notin V$. In case i), we have that $Cn(V \cup \{x\}) = Cn(V) = V$ is a complete set. In case ii), V is a maximal consistent set, in which case $Cn(V \cup \{x\}) = \mathcal{L}$ is a complete set. ■

Theorem 32 $\mathcal{O}_4$ *validates the rules of* $\mathcal{D}_4$ *(for individual formulae a).*

Proof. For SI, assume $x \in \mathcal{O}_4(N, a)$ and $b \vdash a$. Let V be a complete set such that $\{b\} \subseteq V$. We have $Cn(b) \subseteq V$. From the second opening assumption $\{a\} \subseteq Cn(a) \subseteq Cn(b) \subseteq V$. From this and the first opening assumption, $x \in \mathcal{O}_3(N, b)$, by Definition 26. Hence $x \in \mathcal{O}_4(N, b)$, Definition 26 again.

For OR, assume $x \in \mathcal{O}_4(N, a)$ and $x \in \mathcal{O}_4(N, b)$. Let V be a complete set containing $a \vee b$. Since V is complete, either $a \in V$ or $b \in V$. Assume that the first applies. In that case, $x \in \mathcal{O}_3(N, V)$, by the first opening assumption and Definition 26. Assume the second applies. In that case $x \in \mathcal{O}_3(N, V)$, by the second opening assumption and Definition 26. Either way, $x \in \mathcal{O}_3(N, V)$, and thus $x \in \mathcal{O}_4(N, a \vee b)$ as required.

For EQ, assume $x \in \mathcal{O}_4(N, a)$ and $x \dashv\vdash y$. Let V be a complete set containing a. By Definition 26, $x \in \mathcal{O}_3(N, V)$. Trivially, $y \in \mathcal{O}_3(N, V)$, since $x \dashv\vdash y$. So $y \in \mathcal{O}_4(N, a)$ as required.

For ACT, assume $x \in \mathcal{O}_4(N, a)$, $y \in \mathcal{O}_4(N, a \wedge x)$ and $x \wedge y \notin \mathcal{O}_4(N, a)$. From the third opening assumption, there is some complete set V such that $a \in V$ and $x \wedge y \notin \mathcal{O}_3(N, V)$, by Definition 26. Since $a \in V$, the first opening assumption implies $x \in \mathcal{O}_3(N, V)$, by Definition 26 again. Because $\mathcal{O}_3$ satisfies ACT_2 (cf. Remark 24), $y \notin \mathcal{O}_3(N, V')$ where $V' = Cn(V \cup \{x\})$. On the one hand, $a \wedge x \in V'$. On the other hand, by Lemma 31, V' is a complete set. So $y \notin \mathcal{O}_4(N, a \wedge x)$–contradiction. ■

Theorem 33 (Soundness, strong version) $\mathcal{D}_4(N, A) \subseteq \mathcal{O}_4(N, A)$.

Proof. The proof is virtually the same as that for Theorem 8, but using Theorems 28 and 32. ■

Theorem 34 (Completeness, strong version) $\mathcal{O}_4(N, A) \subseteq \mathcal{D}_4(N, A)$.

Proof. The proof is similar to that for Theorem 16. We quickly rerun the verifications for the case where A is a singleton set, still breaking the argument into two cases.

Case 1: a is inconsistent. The argument remains unchanged. Note that the pair $(\bigwedge_{i=1}^n a_i, \bigwedge_{i=1}^n x_i)$ is obtained using AND, which itself follows from ACT.

Case 2: a is consistent. This case may be disposed of using the same kind of maximality argument as in the proof of Theorem 16. Starting with $x \in \mathcal{O}_4(N, a)$ and $x \notin \mathcal{D}_4(N, a)$, one first shows that $\{a\}$ can be extended to some maximal $V \supseteq \{a\}$ such that $x \notin \mathcal{D}_4(N, V)$. One then shows that V is a complete set, and that $x \notin \mathcal{O}_3(N, V)$. (The latter now follows from Theorems 23 and 25: $\mathcal{O}_3(N, V) = \mathcal{D}_3(N, V) \subseteq \mathcal{D}_4(N, V)$.) Definition 26 then yields the required contradiction: $x \notin \mathcal{O}_4(N, a)$. ■

4 Conclusion

This paper has developed variants of the standard I/O operations with the following two salient features. First, they do not satisfy the rule of "weakening the output" (WO). Second, instead of satisfying the traditional rule of "cumulative transitivity" (CT), they satisfy a variant rule called "aggregative cumulative transitivity" (ACT). Each of the proposed variant operations has been given both a semantic characterization and an axiomatic characterization.

We end this paper by listing a number of topics for future research.

First, our variants of out_2 and out_4 have an unexpected feature. We have $out_1(N, A) = Cn(\mathcal{O}_1(N, A))$ and $out_3(N, A) = Cn(\mathcal{O}_3(N, A))$. But we do not have $out_2(N, A) = Cn(\mathcal{O}_2(N, A))$, nor do we have $out_4(N, A) = Cn(\mathcal{O}_4(N, A))$. For a counter-example, take $N = \{(a, x), (b, x \wedge y)\}$ and $A = \{a \vee b\}$. We leave it for future research to define variants of out_2 and out_4 satisfying this property.

Second, we have put to one side the use of constraints, whose aim is to filter out excess output using a consistency check mechanism [27]. Although it remains possible in principle to develop constrained I/O logic on top of the present framework, we still have to investigate the effects of doing it. There is a known connection between the constrained version of I/O logic deployed on top of the standard I/O logics and some well-established non-monotonic formalisms, like Reiter's default logic and the AGM maxi-choice revision operation [1]. This connection was discovered by Makinson and van der Torre [27], with reference to approaches based on so-called belief sets, in which an agent's beliefs are characterized by deductively closed sets of sentences. It would be interesting to know if a similar connection can be made between our framework and variant non-monotonic systems based on so-called belief bases (see, e.g., [11, 29]).

Third, in the papers [36, 38], we have described a family of variant systems with a consistency check built in the semantics and a consistency proviso restraining the application of AND and ACT. The main motivation is the so-called pragmatic oddity [39]. The case of out_4 has not been handled yet.

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