Joining conceptual systems - three remarks on TJS

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Abstract

The Theory of Joining Systems, abbreviated TJS, is a general theory of representing for example legal and other normative systems as formal structures. It uses algebraic tools and a fundamental idea in this algebraic approach is the representation of a conditional norm as an ordered pair of concepts. Another fundamental idea is that the components in such a pair are concepts of different sorts. Conditional norms are thus links from for example descriptive to normative concepts and the result is the joining of two conceptual systems. However, there are often at least three kinds of concepts involved in many normative systems, viz. descriptive, normative and intermediate concepts. Intermediate concepts such as 'being the owner' and 'being a citizen' have descriptive grounds and normative consequences and can be said to be located intermediately between the system of grounds and the system of consequences. Intermediate concepts function as bridges (links, joinings) between concepts of different sorts. The aim of this paper is to further develop TJS and widen the range of application of the theory. It will be shown that the idea of norms as ordered pairs is flexible enough to handle nested implications and hypothetical consequences. Minimal joinings, which are important in TJS, are shown to be closely related to formal concepts in Formal Concept Analysis. TJS was developed for concepts of a special kind, namely conditions. In this paper a new model of TJS is developed, where the concepts are attributes and aspects, and the role of intermediate concepts in this model is discussed.

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1 Outline of the theory of joining systems

1.1 Introduction

Intermediate concepts such as ‘being the owner’, ‘being a guardian’, ‘being a citizen’ and ‘having a relationship similar to being married’ play a significant role in the formulation of legal systems. The meaning and structure of such concepts was the subject of a comprehensive discussion among Scandinavian philosophers and jurists in the 1940’s and 1950’s. The subject has got a renewed interest during the last decades and has been studied from different points of view, for example legal philosophy and logic. In a series of articles further developed and summarized as a chapter in *Handbook of Deontic Logic and Normative Systems* (Lindahl & Odelstad 2013), Lars Lindahl and I present a formal theory of intermediate concepts.[1] This theory is a part of a more general theory of representing for example legal and other normative systems as formal structures. The more general theory is called the Theory of Joining Systems, abbreviated TJS. It uses algebraic tools and a fundamental idea in this algebraic approach is the representation of a conditional norm as an ordered pair of conditions. Another fundamental idea is that the components in such a pair are conditions of different sorts. A simple example is the norm represented by the ordered pair $\langle c_1, c_2 \rangle$ where c_1 is a descriptive and c_2 a normative condition. A conditional norm is thus a link from descriptive conditions to normative conditions.

Intermediate concepts, also called intermediaries, enter the picture as bridges or links between the descriptive and normative conditions. Consequently, there are three kinds of concepts involved in many simple normative systems, descriptive, normative and intermediate conditions. Intermediaries have grounds and consequences, the concept being implied by the grounds and implying the consequences. In simple normative systems an intermediate concept has descriptive grounds and normative consequences and can be said to be located intermediately between the system of grounds and the system of consequences. The intermediate concepts are of another kind than the grounds and consequences, and will here be regarded as belonging to another sort. Therefore, intermediate concepts function as bridges (links, joinings) between concepts of different sorts.

The aim of this paper is to develop TJS in some respects and widen the range of application of the theory. In normative systems it is frequent that the consequence of an intermediary is a conditional norm. By jurists such intermediaries are often said to have hypothetical legal consequences. This means that in norms represented as ordered pairs, one of the components in the pair is itself an ordered pair, and such norms ought to be represented as $\langle c_1, \langle c_2, c_3 \rangle\rangle$ or $\langle\langle c_1, c_2 \rangle, c_3\rangle$. The differences between these two models of representing “norms within norms” and the general treatment of phenomena similar to hypothetical legal consequences within TJS is the subject of one of the remarks in this paper.

An important branch of lattice theory is formal concept analysis. The second remark in this paper focus on a TJS-perspective on what is called formal concepts in this branch of lattice theory. It turns out that the so-called formal concepts are closely

[1] In our chapter of the handbook we discuss other approaches to the problem area.

related to an important notion in TJS, namely the minimal joinings between concepts of different sorts.

The third remark is an introduction to the application of TJS to concepts of quite another kind than conditions, namely the concepts that are often generally called attributes or aspects and in special contexts quantities or qualities. Familiar physical examples are length, mass, force an temperature and examples from other sciences will be discussed in Section 4. Aspects are represented as relational structures and a framework for such representations is presented in subsection 4.2 and 4.3. In subsection 4.4 this framework is used for the construction of an aspect model for TJS.

The logical study on normative systems has applications in many areas, especially in legal science but also for example in computer science and artificial intelligence. There is a discipline emerging on the border between the formal study of normative systems and computer science. This discipline has, at least, a twofold aim: on the one hand a computational approach to normative systems (primarily regarding the law) and on the other hand the study of normative systems used within computer science. This interdisciplinary discipline contains numerous applications of deontic logic and the logic of normative systems. The development of TJS is founded on theoretical considerations but also to some extent with practical applications in view. TJS has for example been used in work on norm-regulation of agent systems, (see Odelstad & Boman, 2004, and Hjelmblom, 2015) on a forest cleaning system (see Odelstad, 2007, and Hjelmblom, 2015, pp. 40–45) and on automation of the Swedish property formation (see Hjelmblom et. al. in press). The development of the aspect model of TJS, which is initialized in this paper, may hopefully result in further applications of TJS

1.2 Some characteristic features of TJS

The description of TJS above is just a first preliminary view of TJS. This section contains a more detailed overview over some aspects of TJS. For a comprehensive presentation, see Lindahl & Odelstad (2013).

TJS is a framework for studying conceptual structures and their relations. The conceptual structures can be of different (logical) types and of different (cognitive) sorts. Essential for the TJS-perspective on conceptual structures is an implicative relation between the concepts. The characteristics of this implicative relation differ depending on the type and sort of the conceptual structure. Examples of two different types of concepts are conditions and attributes (here preferably called aspects). Of special interest from the TJS-perspective is how different conceptual systems are connected to each other. In application of the theory it is frequent that there are many different conceptual structures involved which form a network of different strata. TJS is a theory for the study of many-sorted implicative conceptual systems, *msic-systems* (or *msics*) for short.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two conceptual structures (formally strata) connected by an implicative relation holding between concepts in the two structures. A pair of concepts $\langle c_1, c_2\rangle$, with the first component taken from $\mathcal{A}_1$ and the second from $\mathcal{A}_2$ related by the implicative relation is said to be a joining from $\mathcal{A}_1$ to $\mathcal{A}_2$. A joining can be more or less narrow. Of special interest from a TJS-perspective are the joinings that are maximally narrow. (The exact definition of narrowness is given in Section 2.) In applications of TJS the joining between conceptual structures can represent many different kinds of connections dependent on the type and sort of the structures. The joinings can be norms, rules, scientific laws etc.

A special class of msic-systems is condition implication structures (*cis*). When TJS is applied to such structures the result is in Lindahl & Odelstad (2013) called the *cis model*. Ordinary normative systems consist of condition implication structures and the *cis* model is developed with them in view. Some features of the *cis* model will be listed below (cf. Lindahl & Odelstad, 2013, p. 629f.).

(1) A pair $\langle a_1, a_2 \rangle$ represents a norm due to the normative character of a_2.

(2) The representation aims at a rational reconstruction of a normative system.

(3) Basic entities are concepts (conditions), not sentences or propositions, and the Boolean connectives are in many cases applicable to the conditions, which then constitute a Boolean quasi-ordering.

(4) Emphasis is put on the analysis of minimality of joinings and of closeness between strata.

(5) A central theme is "intermediaries" (intermediate concepts) in the system.

(6) A normative system is represented as a network of subsystems and relations between them; the study comprises stratification of a normative system with structures ("strata") that are intermediate.

(7) Since economy of expression is in focus, representation by a base of minimal joinings is a special interest.

(8) The strata are in many contexts Boolean structures extended with a quasi-ordering (called Boolean quasi-orderings, *Bqo's*). However, the strata of joining-systems need not in the *cis* model be Boolean structures but could instead for example be lattice-like structures.

A note on item (3) above. If p is a ν-ary condition and $i_1,...,i_\nu$ are individuals, then $p(i_1,...,i_\nu)$ is a statement. Conditions can be used in state descriptions of for example social and artificial agent systems. Antecedents and consequences of norms are represented as conditions and are called *grounds* and *consequences* respectively. A norm is correlating a ground to a consequence and is represented as an ordered pair.

Figure 1 is an attempt to illustrate a simple normative system $\mathcal{N}$ consisting of a system $\mathcal{B}_1$ of potential grounds (descriptive conditions) and a system $\mathcal{B}_2$ of potential consequences (normative conditions).[2] The set of norms in $\mathcal{N}$ is the set J of *links* or *joinings* from $\mathcal{B}_1$ to $\mathcal{B}_2$. A norm is represented by an arrow from the system of grounds to the system of consequences.

A norm in a normative system $\mathcal{N}$, the norm here represented as an ordered pair $\langle a_1, a_2 \rangle$, can be regarded as a mechanism of inference. We can distinguish two cases.

[2]The figures in this report are taken from Lindahl & Odelstad (2013) or our other joint publications.

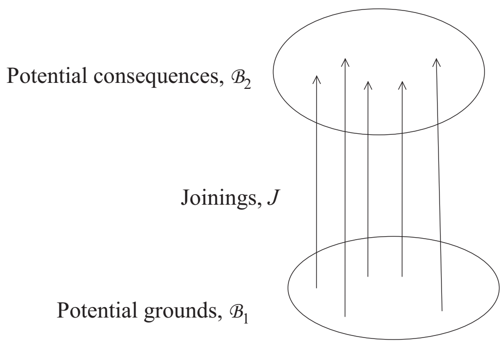

Figure 1: A simple normative system $\mathcal{N}$.

Suppose that a_1 and b_1 are descriptive conditions and a_2 and b_2 normative. $b_1 \precsim_1 a_1$ means that b_1 implies a_1 in $\mathcal{B}_1$ and $a_2 \precsim_2 b_2$ means that a_2 implies b_2 in $\mathcal{B}_2$. Then the following "derivation schemata" are valid given $\mathcal{N}$.

(I)
$a_1(i_1, ..., i_\nu)$
$\langle a_1, a_2 \rangle$

$a_2(i_1, ..., i_\nu)$

(II)
$b_1 \precsim_1 a_1$
$\langle a_1, a_2 \rangle$
$a_2 \precsim_2 b_2$

$\langle b_1, b_2 \rangle$[3]

In (I), $\langle a_1, a_2 \rangle$ functions as a deductive mechanism correlating sentences by means of instantiation, while in (II), $\langle a_1, a_2 \rangle$ plays an important role in correlating one condition, b_1, to another condition, b_2.

A note on item (4) above. Minimality of joinings and of closeness between strata rest on the notion of narrowness between antecedent and consequence in a norm. These notions will be discussed in more detail in later sections, see subsection 1.3.2 and 2.2.1. However, Figure 2 will give a hint of what is meant with narrowness. Consider the norms (links) from the system B_1 of grounds to the system B_2 of consequences.

[3] Note that $b_1 \precsim_1 a_1$ relates conditions of the same sort and the same holds for $a_2 \precsim_2 b_2$; b_1 and a_1 are descriptive but a_2 and b_2 are normative. A norm consists of conditions of different sorts. As stated earlier, only implicative sentences that relate conditions of different sorts will be represented as ordered pairs.

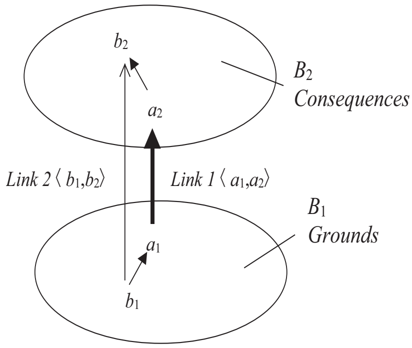

Figure 2: Norm $\langle a_1, a_2\rangle$ is narrower than norm $\langle b_1, b_2\rangle$.

Suppose that $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$ are norms from the system of grounds B_1 to the system of consequences B_2.

The figure illustrates that $\langle a_1, a_2\rangle$ is narrower than $\langle b_1, b_2\rangle$. We can say alternatively that $\langle a_1, a_2\rangle$ "lies between" b_1 and b_2.

A note on item (5). Concepts that have two faces, one turned towards facts and descriptions, the other towards legal consequences are said to be intermediate between facts and legal consequences and will often be called intermediaries. Figure 3 will give an illustration of the idea of a normative system with intermediaries. The system is represented as a two-sorted implicative conceptual system, consisting of a set of descriptive grounds and a set of normative consequences. The intermediate concepts are neither purely descriptive nor purely normative, they have descriptive grounds and normative consequences and must be understood as a unity of the grounds and the consequences.

As an example, consider what it means to be a citizen according to the system of the U.S. Constitution. Article XIV, Section 1 reads as follows:

> All persons born or naturalized in the United States, and subject to the jurisdiction thereof, are citizens of the United States and of the State wherein they reside. No State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States; nor shall any State deprive any person of life, liberty, or property, without the due process of law; nor deny to any person within its jurisdiction the equal protection of the laws.

Two key concepts in the article are *citizen* and *person*. The article specifies the ground for the condition being a citizen in the United States:

persons born or naturalized in the United States, and subject to the jurisdiction thereof

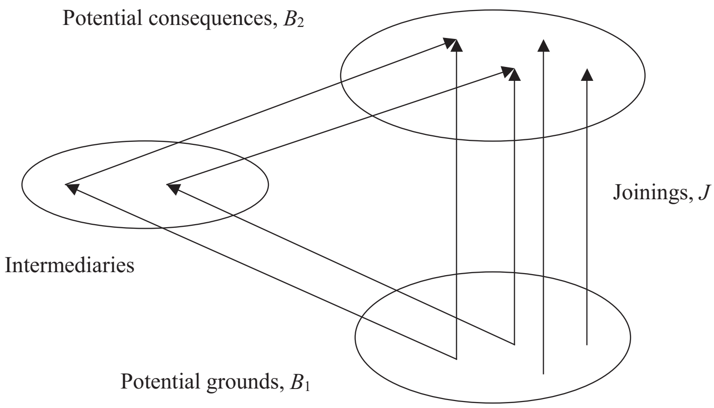

Figure 3: A normative system with intermediaries.

and specifies a number of legal consequences of this condition expressed in terms of 'shall':

no State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States.

The article does not state any ground for the condition to be a person but specifies a number of legal consequences connected to this condition:

nor shall any State deprive any person of life, liberty, or property, without due process of law; nor deny to any person within its jurisdiction the equal protection of the laws.

Within the constitutional system of United States, this article is supplemented with rules laid down by the Constitution and through court decisions. These rules determine together, by specifying grounds and consequences, the role the concepts 'citizen' and 'person' have within the legal system. We will return to this example in Section 2.

A note on item (6). Figure 4 is a network of strata illustrating a TJS representation of a fairly complex normative system. It is included in Lindahl & Odelstad (2013) p. 620 with comments and explanations and these will be quoted here. Note that $\wedge$ is the operation of conjunction and $\vee$ is the operation of disjunction for conditions.

> The present subsection ... presents a *cis* example of joining-systems with intervenients for a network of *Bqo* strata .... The example is legal and concerns *ownership* and *trust* as intervenients. The legal rules in this example are expressed in terms of joinings between *Bqo*'s $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_4$,

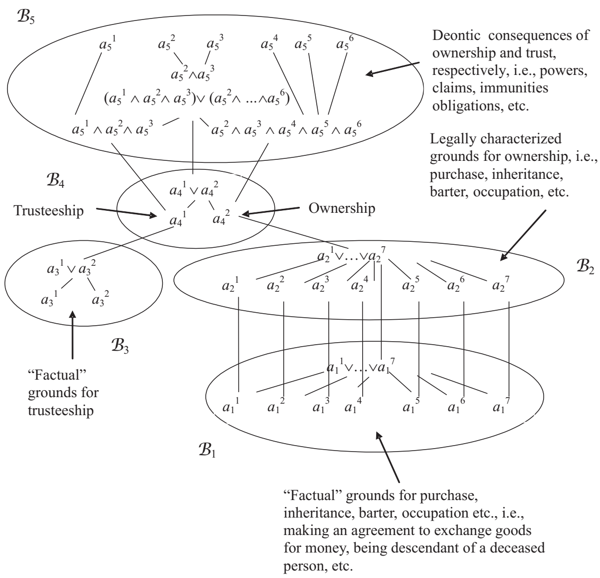

Figure 4: A network of strata.

$\mathcal{B}_5$ for ownership, and between $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{B}_5$ for trusteeship.[4] Both of $\mathcal{B}_2$ and $\mathcal{B}_4$ are intermediate structures, where B_4 is supposed to contain the intervenients ownership and trusteeship and B_2 the intervenients *purchase*, *barter, inheritance, occupation, specification*, *expropriation* (for public purposes or for other reasons), which are grounds for ownership. B_1 contains grounds for the conditions in B_2, such as making a contract for purchase or barter respectively, having particular kinship relationship to a deceased person, appropriating something not owned, creating a valuable thing out of worthless material, getting a verdict on disappropriation of property, either for public purposes or for other reasons. B_3 contains different grounds for trusteeship. B_5 contains the legal consequences of ownership and trusteeship, respectively, in terms of powers, permissions and obligations.

Note that in many (but certainly not all) applications of TJS the conceptual structures involved are of the same type but of different sorts. In the *cis* model the concepts are of the type conditions but that is just one model of TJS. The general, abstract theory of joining-systems can be applied to quasi-orderings of any kind. The “intended models” of TJS differ substantially but have some features in common. They consist

[4] Trust is where a person (trustee) is made the nominal owner of property to be held or used for the benefit of another. Trusteeship is the legal position of a trustee.

of systems of conceptual structures of different sorts with implicative relations joining the structures. In Section 4 we will study *msic*-systems where the concepts are represented as relational structures. Such concepts are often called attributes and sometimes aspects, here the term 'aspect' will be preferred. As examples of aspects the following may be mentioned: area, temperature, age, loudness, accessibility, public interest and archeological value. On concepts of this type several quasi-orderings can be defined, but in this paper only one will be studied.

1.3 A small piece of the TJS formal framework

1.3.1 Introduction

This subsection is an attempt to introduce one basic idea of the TJS-formalism in a very simple fashion.

Let $\langle A_0, \precsim_0\rangle$ be a quasi-ordering. The intended interpretation of A_0 is that its elements are concepts of some type and $\precsim_0$ an implicative relation between the concepts. The character of $\precsim_0$ differ depending on the type of concepts considered. If the concepts are conditions, $\precsim_0$ is usually implication. Define a relation $\trianglelefteq_0$ on $A_0 \times A_0$ as follows:

$$\langle a, b\rangle \trianglelefteq_0 \langle c, d\rangle \Leftrightarrow c \precsim_0 a \;\&\; b \precsim_0 d.$$

In measurement and utility theory $\trianglelefteq_0$ expresses differences. $\langle a, b\rangle \trianglelefteq_0 \langle c, d\rangle$ is then interpreted as follows: The difference (with respect to $\precsim_0$) between a and b is less than or equal to the difference between c and d.

Let A_1 and A_2 be two disjunct subsets of A_0 and let $\precsim_1$ be the restriction of $\precsim_0$ to A_1 and $\precsim_2$ the restriction of $\precsim_0$ to A_2, i.e. $\precsim_1 = \precsim_0 /A_1$ and $\precsim_2 = \precsim_0 /A_2$.[5] We may think of A_1 and A_2 as consisting of concepts of different sorts but (in simple cases) of the same type. For example, the elements in A_1 and A_2 can be conditions but the conditions in A_1 can be descriptive and those in A_2 can be normative. Let $\trianglelefteq$ be restrictions of $\trianglelefteq_0$ to $A_1 \times A_2$, i.e. $\trianglelefteq = \trianglelefteq_0 /A_1 \times A_2$. Note that $\trianglelefteq_0$ is a quasi-ordering on $A_0 \times A_0$ and $\trianglelefteq$ on $A_1 \times A_2$. The elements in $A_1 \times A_2$ are bridges (links, joinings) between the concepts in A_1 and in A_2. In different applications of the theory the ordered pairs in $A_1 \times A_2$ can represent a diversity of phenomena. For example, they can represent conditional norms with the antecedent being a descriptive condition taken from A_1 and the consequence a normative condition taken from A_2. But as will be seen in the last section, the elements in A_1 and A_2 as well as the ordered pairs can represent concepts of quite different types and sorts.

The binary relation $\trianglelefteq$ is a quasi-ordering of the elements in $A_1 \times A_2$. If ordered pairs in $A_1 \times A_2$ represent norms, then $\trianglelefteq$ is an ordering of the norms which can be interpreted as an implicative relation between them. $\langle a, b\rangle \trianglelefteq \langle c, d\rangle$ means that the norm $\langle a, b\rangle$ implies the norm $\langle c, d\rangle$ and we say that $\langle a, b\rangle$ is at least as narrow as $\langle c, d\rangle$. (Cf. differences with respect to $\trianglelefteq_0$.) The minimal norms with respect to $\trianglelefteq$ are of special interest since they can generate the whole set of norms in a transparent way.

$\langle A_0, \precsim_0\rangle$ is in many contexts not given in the outset. Instead, in most cases we start off with two quasi-orderings $\langle A_1, \precsim_1\rangle$ and $\langle A_2, \precsim_2\rangle$. We connect them by pairs $\langle a_1, a_2\rangle$,

[5] Restrictions and the use of '/' is explained in the beginning of Section 1.3.2.

where $a_1 \in A_1$ and $a_2 \in A_2$ and we call them joinings. We close the set of joinings such that it becomes a quasi-ordering with respect to $\trianglelefteq$ and call $\trianglelefteq$ the narrowness-relation from $\langle A_1, \precsim_1 \rangle$ to $\langle A_2, \precsim_2 \rangle$.

Suppose now that we have three quasi-orderings $\langle A_1, \precsim_1 \rangle$, $\langle A_2, \precsim_2 \rangle$ and $\langle A_3, \precsim_3 \rangle$. Suppose further that $\langle a_1, a_2 \rangle$ is a joining from $\langle A_1, \precsim_1 \rangle$ to $\langle A_2, \precsim_2 \rangle$ and $\langle a_2, a_3 \rangle$ is a joining from $\langle A_2, \precsim_2 \rangle$ to $\langle A_3, \precsim_3 \rangle$. If this implies that $\langle a_1, a_3 \rangle$ is a joining from A_1 to A_3 then the elements in A_2 act as intermediaries between A_1 and A_3. Of special interest are those $\langle a_1, a_2 \rangle$ and $\langle a_2, a_3 \rangle$ such that $\langle a_1, a_3 \rangle$ is a minimal joining with respect to the narrowness relation between $\langle A_1, \precsim_1 \rangle$ and $\langle A_3, \precsim_3 \rangle$.

In the next subsection some concepts and results used in TJS will be presented more formally. For a more profound discussion of the subject see Lindahl & Odelstad (2013).

1.3.2 Some basic definitions and results

This section contains definitions and results used in the rest of this paper and is intended to be consulted when necessary. Note that in this paper 'if and only if' is abbreviated to 'iff'.

First a note on terminology. Suppose that R is a ν-ary relation on a set A and that X is a subset of A. Then $R \cap X^\nu$ is denoted R/X and is called the *restriction* of R to X.

Correspondences The notion of a correspondence will be used in Section 3 on Formal Concept Analysis and in Section 4 on joining conceptual systems of aspects.[6]

The triple $\langle X, Y, \gamma \rangle$ is a *correspondence* from X to Y if X and Y are sets, γ is a binary relation, and $\gamma \subseteq X \times Y$. The expressions $\langle x, y \rangle \in \gamma$ and $x\gamma y$ are used synonymously. If $\langle X, Y, \gamma \rangle$ is a correspondence, then

$$\gamma^{-1} = \{\langle y, x \rangle \mid x\gamma y\}$$

and $\left\langle Y, X, \gamma^{-1} \right\rangle$ is a correspondence. If the triple $\langle X, Y, \gamma \rangle$ is a correspondence, it is sometimes more convenient to say that γ is a correspondence from X to Y and that γ^{-1} is a correspondence from Y to X. Suppose that $\langle X, Y, \gamma \rangle$ is a correspondence. If $Z \subseteq X$ we define:

$$\gamma [Z] = \{y \in Y \mid \exists x \in Z : x\gamma y\}.$$

Note that there can exist $Z_1, Z_2 \subseteq X$ such that $Z_1 \neq Z_2$ but $\gamma [Z_1] = \gamma [Z_2]$.

If $W \subseteq Y$ then

$$\gamma^{-1} [W] = \left\{x \in X \mid \exists y \in W : y\gamma^{-1}x\right\} = \{x \in X \mid \exists y \in W : x\gamma y\}.$$

The correspondence $\langle X, Y, \gamma \rangle$ is on X if $\gamma^{-1} [Y] = X$, onto Y if $\gamma [X] = Y$. If a correspondence is on X we say that X is the domain of the correspondence. And if a

[6]The theory of correspondence is frequently used in economic theory, see for example Debreu (1959) and Klein & Thompson (1984), where correspondences are often treated as set-valued functions (see further sub-section 4.3.6 below). The presentation of correspondences is here inspired by Cohn (1965) pp. 9–11.

correspondence is onto Y we say that Y is the image, range or codomain of the correspondence. If there is no risk of ambiguity, we denote $\gamma[\{a\}]$ with $\gamma[a]$ and $\gamma^{-1}[\{b\}]$ with $\gamma^{-1}[b]$.

Suppose that $\langle X, Y, \gamma\rangle$ is a correspondence. Then the set

$$\{\langle x, W\rangle \in X \times \wp(Y) \mid W = \gamma[x]\}$$

is a function on X into $\wp(Y)$, and we denote it $\overrightarrow{\gamma}$. Thus

$$\overrightarrow{\gamma} : X \to \wp(Y)$$

$$\overrightarrow{\gamma}(x) = \gamma[x].$$

The *relative product* of two correspondences $\langle X, Y, \gamma\rangle$ and $\langle Z, W, \delta\rangle$ is the correspondence $\langle X, W, \gamma|\delta\rangle$ where $\gamma|\delta$ is defined by

$$\gamma|\delta = \{\langle x, w\rangle \in X \times W \mid \exists v \in Y \cap Z \in : x\gamma v \;\&\; v\gamma w\}.$$

Note that the operation relative product on correspondences is associative.

Proposition 1 *Suppose that $\langle X, Y, \gamma\rangle$ and $\langle Z, W, \delta\rangle$ are correspondences and that $A \subseteq X$. Then*

$$(\gamma|\delta)[A] = \delta[\gamma[A]].$$

Quasi-orderings

Definition 2 *The binary relation $\precsim$ is a* quasi-ordering on *A if $\precsim$ is transitive and reflexive in A.*

Another name for quasi-ordering is preordering. Writing $\sim$ for the *equality* part of $\precsim$ we say that $x \sim y$ holds iff $x \precsim y$ and $y \precsim x$. Also, writing $\prec$ for the *strict* part of $\precsim$ we say that $x \prec y$ iff $x \precsim y$ and not $y \precsim x$.

A quasi-ordering is closely related to a partial ordering. If $\langle A, \precsim\rangle$ is a quasi-ordering and $\sim$ is the equivalence part of $\precsim$, then $\precsim$ generates a partial ordering on the set of $\sim$-equivalence classes generated from A.

Definition 3 *Suppose that $\precsim$ is a quasi-ordering on A and that $X \subseteq A$ and $x \in X$. Then,*
(1) *x is a* minimal element *in X with respect to $\precsim$ iff there is no $y \in X$ such that $y \prec x$,*
(2) *x is a* maximal element *in X with respect to $\precsim$ iff there is no $y \in X$ such that $x \prec y$.*
(3) *The set of minimal elements in X with respect to $\precsim$ is denoted* $\min_{\precsim} X$ *and the set of maximal elements of X with respect to $\precsim$ is denoted* $\max_{\precsim} X$.
(4) *x is a* least element *in X with respect to $\precsim$ iff for all $y \in X$, $x \precsim y$,*
(5) *x is a* greatest element *in X with respect to $\precsim$ iff for all $y \in X$, $y \precsim x$.*

Note that in a quasi-ordering $\langle A, \precsim\rangle$, a greatest and a least element in a set $X \subseteq A$ need not be unique. But if x and y are greatest elements (or least elements) in X with respect to $\precsim$, then $x \sim y$.

Quasi-lattices and complete quasi-lattices The notions of least upper bound and greatest lower bound are important in the definition of a joining-system. These notions are usually defined for partial orderings and not for quasi-orderings. Since quasi-ordering is a basic structure in TJS, we generalize the notions of least upper bound and greatest lower bound to quasi-orderings. We use ub and lb as abbreviations for upper bound and lower bound respectively, and lub and glb for least upper bound and greatest lower bound respectively. We note that (in contrast to what holds for partial orderings) a least upper bound or a greatest lower bound relative to a quasi-ordering $\langle A, \precsim \rangle$ need not be unique.

Definition 4 *Let $\precsim$ be a quasi-ordering on a set A with $X \subseteq A$. Then*
$\mathrm{ub}_{\precsim} X = \{a \in A \mid \forall x \in X : x \precsim a\}$
$\mathrm{lb}_{\precsim} X = \{a \in A \mid \forall x \in X : a \precsim x\}$
$\mathrm{lub}_{\precsim} X = \left\{a \in A \mid a \in \mathrm{ub}_{\precsim} X \;\&\; \forall b \in \mathrm{ub}_{\precsim} X : a \precsim b\right\}$
$\mathrm{glb}_{\precsim} X = \left\{a \in A \mid a \in \mathrm{lb}_{\precsim} X \;\&\; \forall b \in \mathrm{lb}_{\precsim} X : b \precsim a\right\}.$

According to standard algebraic terminology, a partially ordered set $\langle L, \leq \rangle$ is a lattice if for all $a, b \in L$, $\sup_{\leq} \{a, b\}$ and $\inf_{\leq} \{a, b\}$ exist in L. (In connection with partial orderings, we prefer to use sup and inf instead of lub and glb respectively.) $\langle L, \leq \rangle$ is *complete* if $\inf_{\leq} X$ and $\sup_{\leq} X$ exist for all $X \subseteq L$. We generalize these notions to quasi-orderings.

Definition 5 *If $\langle A, R \rangle$ is a quasi-ordering such that*

$$\mathrm{lub}_R \{a, b\} \neq \varnothing \text{ and } \mathrm{glb}_R \{a, b\} \neq \varnothing \text{ for all } a, b \in A,$$

then $\langle A, R \rangle$ will be called a quasi-lattice. *If* $\mathrm{lub}_R X \neq \varnothing$ *and* $\mathrm{glb}_R X \neq \varnothing$ *for all $X \subseteq A$, then $\langle A, R \rangle$ is a* complete *quasi-lattice.*

Narrowness and lowerness

Definition 6 *Suppose that $\langle A_1, \precsim_1 \rangle$ and $\langle A_2, \precsim_2 \rangle$ are quasi-orderings. The* narrowness relation *with respect to $\langle A_1, \precsim_1 \rangle$ and $\langle A_2, \precsim_2 \rangle$ is a binary relation on $A_1 \times A_2$ denoted by $\trianglelefteq_{\precsim_1, \precsim_2}$, defined as follows: For all $a_1, b_1 \in A_1, a_2, b_2 \in A_2$*

$$\langle a_1, a_2 \rangle \trianglelefteq_{\precsim_1, \precsim_2} \langle b_1, b_2 \rangle \text{ iff } b_1 \precsim_1 a_1 \;\&\; a_2 \precsim_2 b_2. \tag{1}$$

The lowerness relation *with respect to $\langle A_1, \precsim_1 \rangle$ and $\langle A_2, \precsim_2 \rangle$ is a binary relation on $A_1 \times A_2$ denoted by $\precsim^*_{\precsim_1, \precsim_2}$, defined as follows: For all $a_1, b_1 \in A_1, a_2, b_2 \in A_2$*

$$\langle a_1, a_2 \rangle \precsim^*_{\precsim_1, \precsim_2} \langle b_1, b_2 \rangle \text{ iff } a_1 \precsim_1 b_1 \;\&\; a_2 \precsim_2 b_2. \tag{2}$$

When there is no risk of confusion $\trianglelefteq_{1,2}$ will be used instead of $\trianglelefteq_{\precsim_1, \precsim_2}$ and $\precsim^*_{1,2}$ and even $\precsim_{1,2}$ instead of $\precsim^*_{\precsim_1, \precsim_2}$.

Note that $\langle A_1 \times A_2, \trianglelefteq_{1,2} \rangle$ as well as $\left\langle A_1 \times A_2, \precsim^*_{1,2} \right\rangle$ is a quasi-ordering given that $\langle A_1, \precsim_1 \rangle$ and $\langle A_2, \precsim_2 \rangle$ are quasi-orderings.

Protojoining- and joining-systems

Definition 7 *A* protojoining-system (pJs) *is an ordered triple* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *such that* $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ *and* $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ *are quasi-orderings and* $J \subseteq A_1 \times A_2$ *and the following condition is satisfied where* $\trianglelefteq_{1,2}$ *is the narrowness relation determined by* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*: For all* $b_1, c_1 \in A_1$ *and* $b_2, c_2 \in A_2$,

if $\langle b_1, b_2\rangle \in J$ *and* $\langle b_1, b_2\rangle \trianglelefteq_{1,2} \langle c_1, c_2\rangle$*, then* $\langle c_1, c_2\rangle \in J$.

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a protojoining-system we let $\precsim_i$ be the quasi-ordering in $\mathcal{A}_i$ if not stated otherwise.

Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a *pJs*. Then

(1) $\langle A_1, A_2, J\rangle$ is a correspondence with A_1 as domain and A_2 as codomain, and we can also say that J is a correspondence from A_1 to A_2.

(2) $J[A_1] \subseteq A_2$, where $J[A_1]$ contains the second components (belonging to A_2) of the ordered pairs that are joinings from $\mathcal{A}_1$ to $\mathcal{A}_2$, and

(3) $J^{-1}[A_2] \subseteq A_1$, where $J^{-1}[A_2]$ contains the first components (belonging to A_1) of the joinings from $\mathcal{A}_1$ to $\mathcal{A}_{2.}$. Hence, $J^{-1}[A_2]$ is the set of grounds and $J[A_1]$ the set of consequences of the joinings in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$.

(4) $\trianglelefteq_{1,2} [J] \subseteq J$

(5) $\precsim_1 |J| \precsim_2 = J$ and, therefore, J can be said to "absorb" $\precsim_1$ and $\precsim_2$. Note that $x_1(\precsim_1 |J| \precsim_2)x_2$ iff $\exists y_1, y_2 : x_1 \precsim_1 y_1 \;\&\; y_1 J y_2 \;\&\; y_2 \precsim_2 x_2$.

In the next theorem (5) is proved.

Theorem 8 *Suppose that* $\langle A_1, \precsim_1\rangle$ *and* $\langle A_2, \precsim_2\rangle$ *are quasi-orderings and that* $J \subseteq A_1 \times A_2$*. Then* $\langle\langle A_1, \precsim_1\rangle, \langle A_2, \precsim_2\rangle, J\rangle$ *is a* pJs *iff* $\precsim_1 |J| \precsim_2 = J$.

Proof. (I) Suppose that $\langle\langle A_1, \precsim_1\rangle, \langle A_2, \precsim_2\rangle, J\rangle$ is a *pJs*. We prove $\precsim_1 |J| \precsim_2 = J$. Suppose that $\langle a_1, a_2\rangle \in \precsim_1 |J| \precsim_2$. Then there is $b_2 \in A_2$ such that $\langle a_1, b_2\rangle \in (\precsim_1 |J)$ and $b_2 \precsim_2 a_2$. $\langle a_1, b_2\rangle \in (\precsim_1 |J)$ implies that there is $b_1 \in A_1$ such that $a_1 \precsim_1 b_1$ and $\langle b_1, b_2\rangle \in J$. From $a_1 \precsim_1 b_1$ and $b_2 \precsim_2 a_2$ follows that $\langle b_1, b_2\rangle \trianglelefteq_{1,2} \langle a_1, a_2\rangle$ and since $\langle b_1, b_2\rangle \in J$ the definition of a *pJs* implies that $\langle a_1, a_2\rangle \in J$. $a_1 \precsim_1 a_1$, $a_1 J a_2$ and $a_2 \precsim_2 a_2$ implies that $\langle a_1, a_2\rangle \in \precsim_1 |J| \precsim_2$.

(II) Suppose that $\precsim_1 |J| \precsim_2 = J$. Suppose further that $\langle a_1, a_2\rangle \in J$ and $\langle a_1, a_2\rangle \trianglelefteq_{1,2} \langle b_1, b_2\rangle$. It follows that $b_1 \precsim_1 a_1$ and $a_2 \precsim_2 b_2$. $\langle b_1, a_2\rangle \in (\precsim_1 |J)$ and therefore $\langle b_1, b_2\rangle \in (\precsim_1 |J| \precsim_2)$. Hence, $\langle\langle A_1, \precsim_1\rangle, \langle A_2, \precsim_2\rangle, J\rangle$ is a *pJs*. ■

Theorem 9 *Suppose that* $\langle\langle A_1, \precsim_1\rangle, \langle A_2, \precsim_2\rangle, J\rangle$ *is a* pJs*. Then* (1) $\langle J, \trianglelefteq_{1,2} /J\rangle$ *is a quasi-ordering and* (2) $\langle A_1 \cup A_2, (\precsim_1 \cup J \cup \precsim_2)\rangle$ *is a quasi-ordering.*

Proof. (1) $J \subseteq A_1 \times A_2$ and since $\langle A_1 \times A_2, \trianglelefteq_{1,2}\rangle$ is a quasi-ordering it follows that $\langle J, \trianglelefteq_{1,2} /J\rangle$ is a quasi-ordering.

(2) Let $A_0 = A_1 \cup A_2$ and $\precsim_0 = (\precsim_1 \cup J \cup \precsim_2)$. If $x \in A_1$ then $x \sim_1 x$ and if $x \in A_2$ then $x \sim_2 x$. In both cases $x \sim_0 x$ which shows that $\precsim_0$ is reflexive. To prove transitivity suppose that $x \precsim_0 y$ and $y \precsim_0 z$. There are four cases to consider:

(i) $x, y, z \in A_1$ then $x \precsim_1 y$ and $y \precsim_1 z$ and hence $x \precsim_1 z$, which implies that $x \precsim_0 z$.

(ii) $x, y \in A_1$ & $z \in A_2$ and then $x \precsim_1 y$ and yJz and, hence, xJz (according to the theorem above) which implies that $x \precsim_0 z$.

(iii) $x \in A_1$ & $y, z \in A_2$ and then xJy and $y \precsim_2 z$ and, hence, xJz (according to the theorem above) which implies that $x \precsim_0 z$.

(iv) $x, y, z \in A_2$ then $x \precsim_2 y$ and $y \precsim_2 z$ and hence $x \precsim_2 z$, which implies that $x \precsim_0 z$. ■

Definition 10 *A* prejoining-system (preJs)*, is an ordered triple $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ such that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings and $J \subseteq A_1 \times A_2$ and the following conditions are satisfied where $\trianglelefteq_{1,2}$ is the narrowness relation determined by $\mathcal{A}_1$ and $\mathcal{A}_2$:*
(1) *for all $b_1, c_1 \in A_1$ and $b_2, c_2 \in A_2$, if $\langle b_1, b_2\rangle \in J$ and $\langle b_1, b_2\rangle \trianglelefteq_{1,2} \langle c_1, c_2\rangle$, then $\langle c_1, c_2\rangle \in J$,*
(2) *for all $b_1, c_1 \in A_1$ and $b_2 \in A_2$, if $\langle b_1, b_2\rangle \in J$ and $\langle c_1, b_2\rangle \in J$, then $\langle a_1, b_2\rangle \in J$ for all $a_1 \in \mathrm{lub}_{\precsim_1} \{b_1, c_1\}$,*
(3) *for all $b_2, c_2 \in A_2$ and $b_1 \in A_1$, if $\langle b_1, b_2\rangle \in J$ and $\langle b_1, c_2\rangle \in J$, then $\langle b_1, a_2\rangle \in J$ for all $a_2 \in \mathrm{glb}_{\precsim_2} \{b_2, c_2\}$.*

Definition 11 *A* joining-system (Js)*, is an ordered triple $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ such that $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ are quasi-orderings, and $J \subseteq A_1 \times A_2$, and the following conditions are satisfied where $\trianglelefteq_{1,2}$ is the narrowness relation determined by $\mathcal{A}_1$ and $\mathcal{A}_2$:*
(1) *for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, if $\langle a_1, a_2\rangle \in J$ and $\langle a_1, a_2\rangle \trianglelefteq_{1,2} \langle b_1, b_2\rangle$, then $\langle b_1, b_2\rangle \in J$,*
(2) *for any $X_1 \subseteq A_1$ and $a_2 \in A_2$, if $\langle a_1, a_2\rangle \in J$ for all $a_1 \in X_1$, then $\langle b_1, a_2\rangle \in J$ for all $b_1 \in \mathrm{lub}_{\precsim_1} X_1$,*
(3) *for any $X_2 \subseteq A_2$ and $a_1 \in A_1$, if $\langle a_1, a_2\rangle \in J$ for all $a_2 \in X_2$, then $\langle a_1, b_2\rangle \in J$ for all $b_2 \in \mathrm{glb}_{\precsim_2} X_2$.*

(In what follows, when we use the expression $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$, we presuppose that $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$.)

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system, then the elements in J are called *joinings* from $\mathcal{A}_1$ to $\mathcal{A}_2$, and we call J the *joining-space* in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. We call $\mathcal{A}_1$ the *bottom-structure* and $\mathcal{A}_2$ the *top-structure* in the *Js* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$.

In this paper we assume that if $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a *pJs* or a *Js*, then $A_1 \cap A_2 = \varnothing$.

In TJS the notions of connectivity of a joining-system is central.

Minimal Joinings and Connectivity

Definition 12 *A* pJs *$\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ satisfies* connectivity *if whenever $\langle c_1, c_2\rangle \in J$ there is $\langle b_1, b_2\rangle \in J$ such that $\langle b_1, b_2\rangle$ is a minimal element in J with respect to $\trianglelefteq_{1,2}$ and $\langle b_1, b_2\rangle \trianglelefteq_{1,2} \langle c_1, c_2\rangle$.*

The following theorem, which is Theorem 3.26 (p. 579) in Lindahl & Odelstad (2013), gives a sufficient condition for connectivity.

Theorem 13 *If $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ are complete quasi-lattices and $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system, then $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ satisfies connectivity.*

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a *pJs* the set of minimal elements (i.e. joinings) in J is denoted $\min_{\trianglelefteq_{1,2}} J$. Under certain conditions it holds that $\left\langle \min_{\trianglelefteq_{1,2}} J, \precsim^*_{1,2} \right\rangle$ is a complete quasi-lattice, see Corollary 3.36 (p. 588) in Lindahl & Odelstad (2013).

Isomorphisms Suppose that $\mathcal{A} = \langle A, \rho_1, .., \rho_k \rangle$ and $\mathcal{B} = \langle B, \sigma_1, .., \sigma_k \rangle$ are structures such that the arity of ρ_i and σ_i are ν_i for all $i, 1 \leq i \leq k$. A function φ on A onto B is an isomorphism on $\mathcal{A}$ onto $\mathcal{B}$ if for all $i, 1 \leq i \leq k$,

$$\forall a_1, ..., a_{\nu_i} \in A : \rho_i(a_1, ..., a_{\nu_i}) \Leftrightarrow \sigma_i(\varphi(a_1), ..., \varphi(a_{\nu_i})).$$

The following terminology will be used:

$\mathrm{Bi}(A, B)$ the set of bijections (one-to-one correspondences) on the set A onto the set B.

$\mathrm{I}(\mathcal{A}, \mathcal{B})$ the set of isomorphisms on $\mathcal{A}$ onto $\mathcal{B}$.

$\mathrm{I}(\mathcal{A}, \mathcal{A})$ is often shortened to $\mathrm{I}(\mathcal{A})$.

The following proposition, which is easily proven, will be used often and without cross-references in this paper.

Proposition 14 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are structures of the same type and $\varphi \in \mathrm{Bi}(A, B)$. Then*

$$\varphi \in \mathrm{I}(\mathcal{A}, \mathcal{B}) \ \text{ iff } \ \varphi[\mathcal{A}] = \mathcal{B} \ \text{ iff } \ \varphi^{-1}[\mathcal{B}] = \mathcal{A}.$$

2 Hypothetical consequences

2.1 Introduction

Let us construct a simplified "condition-implicative" representation of the legal rules for citizenship in the system of the U.S. Constitution, see subsection 1.2.[7] (Note that $\wedge$ is the operation of conjunction, $\vee$ is the operation of disjunction and $'$ the operation of negation for conditions.) According to the rules, the disjunction of the two conditions

b: to be a person born in the U.S.

n: to be a person naturalized in the U.S.

in conjunction with the condition

s: to be a person subject to the jurisdiction of the U.S.

implies the condition

c: to be a citizen of the U.S.

That this implicative relationship holds according to the system is represented in the form $((b \vee n) \wedge s)\mathcal{R}c$. Since it is a settled matter that citizens who are minors do not have the right to vote in general elections, c does not imply the condition

e: to be entitled to vote in general elections.

Therefore: not $[c\mathcal{R}e]$, and hence not $[((b \vee n) \wedge s)\mathcal{R}e]$.

Let

a: to be adult.

Simplifying matters, suppose that,

(1) $(c \wedge a)\mathcal{R}e$.

It is easy to see that this is equivalent to

(2) $c\mathcal{R}(a' \vee e)$.

[7] The concept citizen regarded as an intermediary is discussed in a number of papers by Lars Lindahl and myself. The presentation here follows mainly Lindahl & Odelstad (2000) pp. 273–277.

Going from (1) to (2) can be called *exportation*, and going from (2) to (1) *importation*.

We thus have within the system the following rules: $((b \vee n) \wedge s)\mathcal{R}c$ and $c\mathcal{R}(a' \vee e)$, stating that the condition $((b \vee n) \wedge s)$ is a ground for c and $(a' \vee e)$ is a consequence of c. These two rules determine partly the role of c (citizenship) in the constitutional system under study. But there can also be other grounds for c and consequences of c within the constitutional system. Suppose that $g_1, g_2, ...$ are the grounds of c and $h_1, h_2, ...$ the consequences of c. Hence, the role of c in the system is characterized by

$$g_1\mathcal{R}c, g_2\mathcal{R}c, ..., c\mathcal{R}h_1, c\mathcal{R}h_2, ...$$

Note that there are several sorts of conditions in this simplified version of the example above. The grounds of c, i.e. b, n, and s, are descriptive, e is normative and c is an intermediary. Let us suppose that the grounds of c belongs to stratum $\mathcal{B}_1$, the intermediary to stratum $\mathcal{B}_2$ and the normative conditions to stratum $\mathcal{B}_3$, and, further, that $J_{1,2}$ is the set of joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$, $J_{2,3}$ the set of joinings from $\mathcal{B}_2$ to $\mathcal{B}_3$ and $J_{1,3}$ the set of joinings from $\mathcal{B}_1$ to $\mathcal{B}_3$. Hence

$$\langle((b \vee n) \wedge s), c\rangle \in J_{1,2}.$$

However, note that

$$\langle c, (a' \vee e)\rangle \notin J_{2,3}$$

since $(a' \vee e)$ is a mixture of two sorts, descriptive and normative, and belongs neither to $\mathcal{B}_1$ nor to $\mathcal{B}_3$. To go from $\langle c, (a' \vee e)\rangle$ to $\langle(c \wedge a), e\rangle$ by importation does not solve the problem, since we get $(c \wedge a)$ which is a mixture of an intermediary and a descriptive condition. A solution seems to be to construct a "mixed stratum" $\mathcal{B}_1$ and $\mathcal{B}_2$ so that $(c \wedge a)$ belongs to that. But that procedure undermines the idea that an intermediary is implied by its grounds and implies its consequences. However, there is another possibility worth considering. Note that $(a' \vee e)$ can be represented as a norm $a\mathcal{R}e$ and we can express $c\mathcal{R}(a' \vee e)$ as the "nested implication" $c\mathcal{R}(a\mathcal{R}e)$. We can represent this implication as the ordered pair $\langle c, \langle a, e\rangle\rangle$ that contains an ordered pair as one of its components; $\langle a, e\rangle$ is a norm within a norm. Note that a is a descriptive condition but of another sort than b, n, and s, since a is not a ground for c, instead we may suppose that a belongs to the stratum $\mathcal{B}_4$. We thus have

$$\langle((b \vee n) \wedge s), c\rangle \in J_{1,2},\ \langle a, e\rangle \in J_{4,3}\ \langle c, \langle a, e\rangle\rangle \in J_{2,(4,3)}.$$

The consequence of c is hypothetical, since it is conditional on being adult. That the consequence of a condition, for example an intermediary, is itself a norm is a phenomenon of frequent occurrence in law, and jurists often call such consequences "hypothetical legal consequences".

2.2 Formal treatment

In this subsection we will investigate the relation between representing the same norm either as $\langle\langle c_1, c_2\rangle, c_3\rangle$ or $\langle c_1, \langle c_2, c_3\rangle\rangle$, and in that way hopefully contribute to the discussion of hypothetical legal consequences. The following abbreviations will be used: '*Js*' for joining-system, '*Js(i)*' for the i:th condition in the definition of a joining-system and '*pJs*' for protojoining-system.

2.2.1 Narrowness and lowerness

Suppose that $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$, $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ and $\mathcal{A}_3 = \langle A_3, \precsim_3\rangle$ are quasi-orderings. We infer the following ordering relations.

$\trianglelefteq_{i,j}$, also denoted $\trianglelefteq_{\precsim_i,\precsim_j}$, is the narrowness condition on $A_i \times A_j$ determined by $\precsim_i$ and $\precsim_j$.

$\precsim^*_{i,j}$, also denoted $\precsim_{\precsim_i,\precsim_j}$, is the lowerness condition on $A_i \times A_j$ determined by $\precsim_i$ and $\precsim_j$.

$\trianglelefteq_{1,(2,3)}$, also denoted $\trianglelefteq_{\precsim_1,\trianglelefteq_{\precsim_2,\precsim_3}}$ or $\trianglelefteq_{\precsim_1,\trianglelefteq_{2,3}}$, is the narrowness relation on $A_1\times(A_2 \times A_3)$ determined by $\precsim_1$ and $\trianglelefteq_{2,3}$.

$\trianglelefteq_{(1,2)^*,3}$, also denoted $\trianglelefteq_{\precsim^*_{\precsim_1,\precsim_2},\precsim_3}$ or $\trianglelefteq_{\precsim^*_{1,2},\precsim_3}$, is the narrowness relation on $(A_1 \times A_2)\times A_3$ determined by $\precsim^*_{1,2}$ and $\precsim_3$.

The following structures are quasi-orderings: $\left\langle A_i \times A_j, \trianglelefteq_{i,j}\right\rangle$, $\left\langle A_i \times A_j, \precsim^*_{i,j}\right\rangle$, $\langle A_1 \times (A_2 \times A_3), \trianglelefteq_{1,(2,3)}\rangle$ and $\langle (A_1 \times A_2) \times A_3, \trianglelefteq_{(1,2)^*,3}\rangle$.

The equality part of $\trianglelefteq$ will be denoted $\backsimeq$ (with appropriate index).

Note the following:

$$\langle\langle a_1, a_2\rangle, a_3\rangle \trianglelefteq_{(1,2)^*,3} \langle\langle b_1, b_2\rangle, b_3\rangle \Leftrightarrow \langle b_1, b_2\rangle \precsim^*_{1,2} \langle a_1, a_2\rangle \;\&\; a_3 \precsim_3 b_3 \Leftrightarrow$$
$$\Leftrightarrow b_1 \precsim_1 a_1 \;\&\; b_2 \precsim_2 a_2 \;\&\; a_3 \precsim_3 b_3.$$

And further:

$$\langle a_1, \langle a_2, a_3\rangle\rangle \trianglelefteq_{1,(2,3)} \langle b_1, \langle b_2, b_3\rangle\rangle \Leftrightarrow b_1 \precsim_1 a_1 \;\&\; \langle a_2, a_3\rangle \trianglelefteq_{2,3} \langle b_2, b_3\rangle \Leftrightarrow$$
$$\Leftrightarrow b_1 \precsim_1 a_1 \;\&\; b_2 \precsim_2 a_2 \;\&\; a_3 \precsim_3 b_3.$$

Hence,

$$\langle\langle a_1, a_2\rangle, a_3\rangle \trianglelefteq_{(1,2)^*,3} \langle\langle b_1, b_2\rangle, b_3\rangle \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \trianglelefteq_{1,(2,3)} \langle b_1, \langle b_2, b_3\rangle\rangle.$$

More exhaustively it can be expressed

$$\langle\langle a_1, a_2\rangle, a_3\rangle \trianglelefteq_{\precsim^*_{\precsim_1,\precsim_2},\precsim_3} \langle\langle b_1, b_2\rangle, b_3\rangle \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \trianglelefteq_{\precsim_1,\trianglelefteq_{\precsim_2,\precsim_3}} \langle b_1, \langle b_2, b_3\rangle\rangle.$$

Theorem 15 *Suppose that*

$$\varphi : (A_1 \times A_2) \times A_3 \to A_1 \times (A_2 \times A_3)$$

such that

$$\varphi(\langle a_1, a_2\rangle, a_3) = \langle a_1, \langle a_2, a_3\rangle\rangle.$$

Then φ is an isomorphism on $\langle (A_1 \times A_2) \times A_3, \trianglelefteq_{(1,2)^*,3}\rangle$ onto $\langle A_1 \times (A_2 \times A_3), \trianglelefteq_{1,(2,3)}\rangle$.

Proof. First note that φ is a bijection on $(A_1 \times A_2) \times A_3$ onto $A_1 \times (A_2 \times A_3)$. Further,

$$\langle\langle a_1, a_2\rangle, a_3\rangle \trianglelefteq_{(1,2)^*,3} \langle\langle b_1, b_2\rangle, b_3\rangle \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \trianglelefteq_{1,(2,3)} \langle b_1, \langle b_2, b_3\rangle\rangle \Leftrightarrow$$
$$\varphi(\langle a_1, a_2\rangle, a_3) \trianglelefteq_{1,(2,3)} \varphi(\langle b_1, b_2\rangle, b_3).$$

This shows the theorem. ■

2.2.2 Protojoining-systems

Theorem 16 *Suppose* $J_{(1,2),3} \subseteq (A_1 \times A_2) \times A_3$ *and* $J_{1,(2,3)} \subseteq A_1 \times (A_2 \times A_3)$ *such that*

$$\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3} \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}. \tag{JJ}$$

Then
(1)
$\left\langle\left\langle A_1 \times A_2, \precsim^*_{1,2}\right\rangle, \langle A_3, \precsim_3\rangle, J_{(1,2),3}\right\rangle$ *is a* pJs $\Leftrightarrow \langle\langle A_1, \precsim_1\rangle, \langle A_2 \times A_3, \trianglelefteq_{2,3}\rangle, J_{1,(2,3)}\rangle$ *is a* pJs.
(2)
φ *such that* $\varphi(\langle a_1, a_2\rangle, a_3) = \langle a_1, \langle a_2, a_3\rangle\rangle$ *is an isomorphism on* $\langle J_{(1,2),3}, \trianglelefteq_{(1,2)^*,3}\rangle$ *onto* $\langle J_{1,(2,3)}\ \trianglelefteq_{1,(2,3)}\rangle$.

Proof. (1) We prove ($\Rightarrow$). Suppose that $\left\langle\left\langle A_1 \times A_2, \precsim^*_{1,2}\right\rangle, \langle A_3, \precsim_3\rangle, J_{(1,2),3}\right\rangle$ is a *pJs*. Suppose further that $a_1, b_1 \in A_1$, $\langle a_2, a_3\rangle, \langle b_2, b_3\rangle \in A_2 \times A_3$, $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ and that the following holds:

$$\langle a_1, \langle a_2, a_3\rangle\rangle \trianglelefteq_{1,(2,3)} \langle b_1, \langle b_2, b_3\rangle\rangle.$$

Then $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A_1 \times A_2, a_3 \in A_3, \langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ and

$$\langle\langle a_1, a_2\rangle, a_3\rangle \trianglelefteq_{(1,2)^*,3} \langle\langle b_1, b_2\rangle, b_3\rangle.$$

Since $\left\langle\left\langle A_1 \times A_2, \precsim^*_{1,2}\right\rangle, \langle A_3, \precsim_3\rangle, J_{(1,2),3}\right\rangle$ is a *pJs* then $\langle\langle b_1, b_2\rangle, b_3\rangle \in J_{(1,2),3}$. Hence, $\langle b_1, \langle b_2, b_3\rangle\rangle \in J_{1,(2,3)}$ which shows $\langle\langle A_1, \precsim_1\rangle, \langle A_2 \times A_3, \trianglelefteq_{2,3}\rangle, J_{1,(2,3)}\rangle$ is a *pJs*. The other direction, i.e. ($\Leftarrow$), of the equivalence in (1) is proved analogously.

(2) According to Theorem 15, φ is an isomorphism on $\langle (A_1 \times A_2) \times A_3, \trianglelefteq_{(1,2)^*,3}\rangle$ onto

$$\langle A_1 \times (A_2 \times A_3), \trianglelefteq_{1,(2,3)}\rangle.$$

Note that $\langle J_{(1,2),3}, \trianglelefteq_{(1,2)^*,3}\rangle$ is a substructure of $\langle (A_1 \times A_2) \times A_3, \trianglelefteq_{(1,2)^*,3}\rangle$ and $\langle J_{(1,2),3}, \trianglelefteq_{(1,2)^*,3}\rangle$ is a substructure of $\langle A_1 \times (A_2 \times A_3), \trianglelefteq_{1,(2,3)}\rangle$ and, furthermore,

$$\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3} \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)} \Leftrightarrow \varphi(\langle a_1, a_2\rangle, a_3) \in J_{(1,2),3}.$$

■

2.2.3 Equivalence results

For simplification of notation let

$$\mathcal{A}_{1,(2,3)} = \langle\langle A_1, \precsim_1\rangle, \langle A_2 \times A_3, \trianglelefteq_{2,3}\rangle, J_{1,(2,3)}\rangle$$
$$\mathcal{A}_{(1,2),3} = \left\langle\left\langle A_1 \times A_2, \precsim^*_{1,2}\right\rangle, \langle A_3, \precsim_3\rangle, J_{(1,2),3}\right\rangle$$

In this subsection we suppose that $\mathcal{A}_{(1,2),3}$ and thus also $\mathcal{A}_{1,(2,3)}$ are *pJs's*. Hence, $\langle J_{(1,2),3}, \trianglelefteq_{(1,2)^*,3}\rangle$ and $\langle J_{1,(2,3)}, \trianglelefteq_{1,(2,3)}\rangle$ are quasi-orderings. We also suppose that (JJ) in Theorem 16 holds. φ is defined as in Theorem 15.

In the study of *pJs's* the minimal joinings play a special role for characterizing the system (see subsection 1.3.2). The following theorem and its corollary are therefore of interest.

Theorem 17 *For all* $a_1 \in A_1$, $a_2 \in A_2$ *and* $a_3 \in A_3$

$$\langle\langle a_1, a_2\rangle, a_3\rangle \in \min J_{(1,2),3} \Leftrightarrow \langle a_1, \langle a_2, a_3\rangle\rangle \in \min J_{1,(2,3)}.$$

Proof. We prove ($\Rightarrow$). Suppose that $\langle\langle a_1, a_2\rangle, a_3\rangle \in \min J_{(1,2),3}$ and, hence, $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$. Suppose now that $\langle b_1, \langle b_2, b_3\rangle\rangle \in J_{1,(2,3)}$ such that

$$\langle b_1, \langle b_2, b_3\rangle\rangle \trianglelefteq_{1,(2,3)} \langle a_1, \langle a_2, a_3\rangle\rangle.$$

Then $\langle\langle b_1, b_2\rangle, b_3\rangle \in J_{(1,2),3}$ and

$$\langle\langle b_1, b_2\rangle, b_3\rangle \trianglelefteq_{(1,2),3} \langle\langle a_1, a_2\rangle, a_3\rangle$$

and since $\langle\langle a_1, a_2\rangle, a_3\rangle \in \min J_{(1,2),3}$ it follows that

$$\langle\langle b_1, b_2\rangle, b_3\rangle \backsimeq_{(1,2),3} \langle\langle a_1, a_2\rangle, a_3\rangle$$

which implies that

$$\langle b_1, \langle b_2, b_3\rangle\rangle \backsimeq_{1,(2,3)} \langle a_1, \langle a_2, a_3\rangle\rangle$$

and, hence, $\langle a_1, \langle a_2, a_3\rangle\rangle \in \min J_{1,(2,3)}$. The proof of ($\Leftarrow$) is analogous. ■

Corollary 18 $\mathcal{A}_{(1,2),3}$ *satisfies connectivity iff* $\mathcal{A}_{1,(2,3)}$ *satisfies connectivity.*

Theorem 16, Theorem 17 and Corollary 18 show in what sense there is an equivalence between representing a norm as $\langle\langle c_1, c_2\rangle, c_3\rangle$ or $\langle c_1, \langle c_2, c_3\rangle\rangle$. Note that it is of course possible that c_3 is itself an ordered pair (for example a norm) $\langle d_1, d_2\rangle$ and we get $\langle\langle c_1, c_2\rangle, \langle d_1, d_2\rangle\rangle$ and $\langle c_1, \langle c_2, \langle d_1, d_2\rangle\rangle\rangle$ respectively. This process can be iterated.

2.2.4 Some theorems

Even if $\mathcal{A}_{1,(2,3)}$ and $\mathcal{A}_{(1,2),3}$ are *pJs's* and φ an isomorphism on $\langle J_{(1,2),3}, \trianglelefteq_{(1,2)^*,3}\rangle$ onto $\langle J_{1,(2,3)}, \trianglelefteq_{1,(2,3)}\rangle$ it does not seem to follow that $\mathcal{A}_{1,(2,3)}$ and $\mathcal{A}_{(1,2),3}$ share the same classification with regard to being a *Js*. Note for example that if $\mathcal{A}_{1,(2,3)}$ is a *Js* and the quasi-orderings in the system are complete quasi-lattices, then $\mathcal{A}_{1,(2,3)}$ satisfies connectivity, which implies that this holds of $\mathcal{A}_{(1,2),3}$, too. But it does not seem to follow that $\mathcal{A}_{(1,2),3}$ is a *Js*. The situation is complicated, which is illustrated by the following theorems. (This subsection can be omitted without loss of continuity.) In the next three theorems we make assumptions about $\mathcal{A}_{1,(2,3)}$ and examine the result of these for $\mathcal{A}_{(1,2),3}$. In the last two theorems in this subsection we make assumptions about $\mathcal{A}_{(1,2),3}$ and examine the result of these for $\mathcal{A}_{1,(2,3)}$.

Theorem 19 *Suppose that* $\mathcal{A}_{1,(2,3)}$ *is a* pJs *such that* Js(3) *is satisfied and that* (JJ) *holds. Suppose further that* $X_2 \subseteq A_2$, $a_1 \in A_1$, $a_3 \in A_3$ *and* $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $a_2 \in X_2$. *If* $\text{glb}_{\trianglelefteq_{2,3}}(X_2 \times \{a_3\}) \neq \varnothing$ *then* $\langle\langle a_1, b_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $b_2 \in \text{lub}_{\precsim_2} X_2$.[8]

[8]Note that the condition $\text{glb}_{\trianglelefteq_{2,3}}(X_2 \times \{a_3\}) \neq \varnothing$ is satisfied if $\langle A_1 \times A_2, \trianglelefteq_{1,2}\rangle$ is a complete quasi-lattice (see Lindahl & Odelstad, 2013, p. 568).

Proof. Given the assumptions in the theorem we can proceed as follows. $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $a_2 \in X_2$. Since $\mathcal{A}_{1,(2,3)}$ satisfies condition *Js(3)* it follows that $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $\langle a_2, a_3\rangle \in X_2 \times \{a_3\}$ implies that $\langle a_1, \langle c_2, c_3\rangle\rangle \in J_{1,(2,3)}$ for all $\langle c_2, c_3\rangle \in \text{glb}_{\trianglelefteq_{2,3}} (X_2 \times \{a_3\})$. Suppose that $b_2 \in \text{lub}_{\precsim_2} X_2$ and $x_2 \in X_2$. Then $x_2 \precsim_2 b_2$ and, hence,

$$\langle b_2, a_3\rangle \trianglelefteq_{2,3} \langle x_2, a_3\rangle$$

from which follows that $\langle b_2, a_3\rangle \in \text{lb}_{\trianglelefteq_{2,3}} (X_2 \times \{a_3\})$.
Suppose that $\langle c_2, c_3\rangle \in \text{glb}_{\trianglelefteq_{2,3}} (X_2 \times \{a_3\})$. Then it follows that

$$(1) \quad \langle b_2, a_3\rangle \trianglelefteq_{2,3} \langle c_2, c_3\rangle$$

which implies $c_2 \precsim_2 b_2$ and $a_3 \precsim_3 c_3$, and, furthermore, it follows that

$$(2) \quad \langle c_2, c_3\rangle \trianglelefteq_{2,3} \langle x_2, a_3\rangle$$

for all $y_2 \in X_2$, which implies $y_2 \precsim_2 c_2$ and $c_3 \precsim_3 a_3$. Hence, $c_2 \in \text{ub}_{\precsim_2} X_2$ which together with $b_2 \in \text{lub}_{\precsim_2} X_2$ implies $b_2 \precsim_2 c_2$. We have thus shown that $c_2 \sim_2 b_2$ and $c_3 \sim_3 a_3$, and thus $\langle b_2, a_3\rangle \in \text{glb}_{\trianglelefteq_{2,3}} (X_2 \times \{a_3\})$. From condition *Js(3)* it follows that $\langle a_1, \langle b_2, a_3\rangle\rangle \in J_{1,(2,3)}$ which implies $\langle\langle a_1, b_2\rangle, a_3\rangle \in J_{(1,2),3}$. ■

Theorem 20 *Suppose that* $\mathcal{A}_{1,(2,3)}$ *is a* pJs *such that* Js(2) *is satisfied and that* (JJ) *holds. Suppose further that* $X_1 \subseteq A_1, a_2 \in A_2, a_3 \in A_3$ *and* $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $a_1 \in X_1$. *Then* $\langle\langle b_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $b_1 \in \text{lub}_{\precsim_1} X_1$.

Proof. Given the assumptions in the theorem we can proceed as follows. From $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ and (JJ) follows that $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$, and given the assumptions in the theorem this holds for all $a_1 \in X_1$. Then, according to condition *Js(2)* applied to $\mathcal{A}_{1,(2,3)}$, $\langle b_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $b_1 \in \text{lub}_{\precsim_1} X_1$. Hence, $\langle\langle b_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ for all $b_1 \in \text{lub}_{\precsim_1} X_1$. ■

Theorem 21 *Suppose that* $\mathcal{A}_{1,(2,3)}$ *is a* pJs *such that* Js(3) *is satisfied and that* (JJ) *holds. Suppose further that* $X_3 \subseteq A_3, a_1 \in A_1, a_2 \in A_2$ *and* $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $a_3 \in X_3$. *Then the following holds: If* $\text{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3) \neq \varnothing$ *then* $\langle\langle a_1, a_2\rangle, b_3\rangle \in J_{(1,2),3}$ *for all* $b_3 \in \text{glb}_{\precsim_3} X_3$.[9]

Proof. Given the assumptions in the theorem we can proceed as follows. $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $a_3 \in X_3$, i.e. $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $\langle a_2, a_3\rangle \in \{a_2\} \times X_3$. Since $\mathcal{A}_{1,(2,3)}$ satisfies condition *Js(3)* it follows that $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $\langle a_2, a_3\rangle \in \{a_2\} \times X_3$ implies that $\langle a_1, \langle c_2, c_3\rangle\rangle \in J_{1,(2,3)}$ for all $\langle c_2, c_3\rangle \in \text{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3)$. Suppose that $b_3 \in \text{glb}_{\precsim_3} X_3$. Then $b_3 \precsim_3 x_3$ for all $x_3 \in X_3$ and hence

$$\langle a_2, b_3\rangle \trianglelefteq_{2,3} \langle a_2, x_3\rangle .$$

This shows that $\langle a_2, b_3\rangle \in \text{lb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3)$. Suppose that

$$\langle y_2, y_3\rangle \in \text{lb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3) .$$

[9]Note that the condition $\text{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3) \neq \varnothing$ is satisfied if $\langle A_1 \times A_2, \trianglelefteq_{1,2}\rangle$ is a quasi-lattice (see Lindahl & Odelstad, 2013, p. 568).

Then for all $x_3 \in X_3$

$$\langle y_2, y_3\rangle \trianglelefteq_{2,3} \langle a_2, x_3\rangle$$

which implies that $a_2 \precsim_2 y_2$ and $y_3 \precsim_2 x_3$. It follows that $y_3 \in \mathrm{lb}_{\precsim_3} X_3$ and since $b_3 \in \mathrm{glb}_{\precsim_3} X_3$ it follows that $y_3 \precsim_2 b_3$. Suppose that $\langle y_2, y_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3)$. Then

$$\langle a_2, b_3\rangle \trianglelefteq_{2,3} \langle y_2, y_3\rangle .$$

Then $y_2 \precsim_2 a_2$ and $b_3 \precsim_3 y_3$. It follows that $y_2 \sim_2 a_2$ and $y_3 \sim_3 y_3$ and, hence, $\langle a_2, b_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3)$. From condition *Js(3)* it follows that $\langle a_1, \langle a_2, b_3\rangle\rangle \in J_{1,(2,3)}$ which implies $\langle\langle a_1, a_2\rangle, b_3\rangle \in J_{(1,2),3}$. ■

Theorem 22 *Suppose that* $\mathcal{A}_{(1,2),3}$ *is a* pJs *and that* (JJ) *holds. Suppose further that* if (1) $X_1 \subseteq A_1$, (2) $a_2 \in A_2, a_3 \in A_3$ *and* (3) $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $a_1 \in X_1$, then $\langle\langle b_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ *for all* $b_1 \in \mathrm{lub}_{\precsim_1} X_1$. *Then* $\mathcal{A}_{1,(2,3)}$ *satisfies* Js(2)*.*

Proof. Given the assumptions in the theorem we can proceed as follows: Suppose that $X_1 \subseteq A_1$, (2) $a_2 \in A_2, a_3 \in A_3$ and (3) $\langle a_1, \langle a_2, b_3\rangle\rangle \in J_{1,(2,3)}$ for all $a_1 \in X_1$. Then $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ and according to the assumptions in the theorem it follows that $\langle\langle b_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ for all $b_1 \in \mathrm{lub}_{\precsim_1} X_1$. Hence, $\langle a_1, \langle a_2, b_3\rangle\rangle \in J_{1,(2,3)}$ for all $b_1 \in \mathrm{lub}_{\precsim_1} X_1$, which shows that $\mathcal{A}_{1,(2,3)}$ satisfies *Js(2)*. ■

Theorem 23 *Suppose that* $\mathcal{A}_{(1,2),3}$ *is a* pJs *such that* Js(3) *is satisfied and that* (JJ) *holds. Suppose further that* $X_3 \subseteq A_3, a_1 \in A_1, a_2 \in A_2$ *and* $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ *for all* $a_3 \in X_3$. *Then the following holds: If* $\mathrm{glb}_{\trianglelefteq_{2,3}} (\{a_2\} \times X_3) \neq \varnothing$ *then* $\langle a_1, \langle a_2, b_3\rangle\rangle \in J_{1,(2,3)}$ *for all* $\langle a_2, b_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$.

Proof. Note first that $\mathcal{A}_{(1,2),3}$ satisfies *Js(3)* iff the following holds: *If* (1) $X_3 \subseteq A_3$, (2) $a_1 \in A_1, a_2 \in A_2$ and (3) $\langle\langle a_1, a_2\rangle, a_3\rangle \in J_{(1,2),3}$ for all $a_3 \in X_3$ *then* $\langle\langle a_1, a_2\rangle, b_3\rangle \in J_{(1,2),3}$ for all $b_3 \in \mathrm{glb}_{\precsim_3} X_3$. Suppose that $X_3 \subseteq A_3, a_1 \in A_1, a_2 \in A_2$ and $\langle a_1, \langle a_2, a_3\rangle\rangle \in J_{1,(2,3)}$ for all $a_3 \in X_3$. From this follows that $\langle\langle a_1, a_2\rangle, b_3\rangle \in J_{(1,2),3}$ for all $b_3 \in \mathrm{glb}_{\precsim_3} X_3$. We now show that if $b_3 \in \mathrm{glb}_{\precsim_3} X_3$ then $\langle a_2, b_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$. Suppose that $b_3 \in \mathrm{glb}_{\precsim_3} X_3$ and that $x_3 \in X_3$. Then $b_3 \precsim_3 x_3$ and

$$\langle a_2, b_3\rangle \trianglelefteq_{2,3} \langle a_2, x_3\rangle$$

from which follows that $\langle a_2, b_3\rangle \in \mathrm{lb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$. Suppose that $\langle c_2, c_3\rangle \in \mathrm{lb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$. Then

$$\langle c_2, c_3\rangle \trianglelefteq_{2,3} \langle a_2, x_3\rangle$$

and, hence, $a_2 \precsim_2 c_2$ and $c_3 \precsim_3 x_3$ and hence $c_3 \in \mathrm{lb}_{\precsim_3} X_3$ which implies $c_3 \precsim_3 b_3$. Suppose further that $\langle c_2, c_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$. Then

$$\langle a_2, b_3\rangle \trianglelefteq_{2,3} \langle c_2, c_3\rangle$$

from which follows that $c_2 \precsim_2 a_2$ and $b_3 \precsim_3 c_3$. Hence, $c_2 \sim_2 a_2$ and $b_3 \sim_3 c_3$ which shows that $\langle a_2, b_3\rangle \in \mathrm{glb}_{\trianglelefteq_{2,3}} \{a_2\} \times X_3$. ■

The above theorems suggest that the relation between the formal properties of systems consisting of three sorts of concepts and used for representing norms as $\langle\langle c_1, c_2\rangle, c_3\rangle$ or $\langle c_1, \langle c_2, c_3\rangle\rangle$ is somewhat complicated. A complete analysis of this problem area deserves a more thorough treatment than what is given here.

3 Formal Concept Analysis and TJS

3.1 Introduction

Formal Concept Analysis (FCA for short) is a field of applied mathematics, chiefly a branch of applied lattice theory. The topic was introduced by Rudolf Wille in 1982 and an excellent presentation of the field is given in Ganter & Wille (1999). A brief introduction to Formal Concept Analysis is given in Davey & Priestley (2002) Chapter 3. In the first chapter of their book Davey and Priestley give the following very brief description of the subject.

> . . . the rather new discipline of **concept analysis** provides a powerful technique for classifying and for analysing complex sets of data. From a set of objects (to take a simple example, the planets) and a set of attributes (for the planets, perhaps large/small, moon/no moon, near sun/far from sun), concept analysis builds an ordered set which reveals inherent hierarchical structure and thence natural groupings and dependencies among the objects and the attributes. (Davey and Priestly, 2002, p. 6)

In an interesting paper Audun Stolpe sets out to study the particular tangential point which exists between input/output logic and FCA (see Stolpe, 2015). As Stolpe points out (p. 240):

> Since the set of axioms in any given input/output logic is just a binary relation between formulae, it ought to be possible to apply results from FCA to the study of forms of conditionality that are not naturally assimilated to the model based on inference relations and/or conditionals–e.g. to sets of norms . . .

Stolpe uses FCA for providing a semantics for input/output logic. Since there are some striking similarities (but also differences) between input/output logic (see Lindahl & Odelstad 2013 pp. 627–631, Stolpe pp. 256–257 and Sun 2013) it is a reasonable conjecture that formal concepts in the sense of Wille can be a useful tool in TJS. The theorems below throw some light on this topic. It is important to note that the expression '[formal] concept' as it is used in FCA does not have the same meaning as the word 'concept' as that term is used in TJS.

Two central notions in FCA are 'context' and 'concept'. With the notation used in Davey & Priestley (2002) they can be introduced as follows. A *context* is a triple (G, M, I) where G and M are sets and $I \subseteq G \times M$. The elements of G and M are called *objects* and *attributes* respectively.

Let $A \subseteq G$ and $B \subseteq M$. Then

$$A' =_{df} \{m \in M \mid (\forall g \in A)\ gIm\}$$

$$B' =_{df} \{g \in G \mid (\forall m \in B)\ gIm\}.$$

Let (G, M, I) be a *context*, $A \subseteq G$ and $B \subseteq M$. Then (A, B) is a *concept* of (G, M, I) iff $A' = B$ and $B' = A$. The set of all concepts of the context (G, M, I) is denoted $\mathfrak{B}(G, M, I)$.

A basic ingredient in FCA is thus a context consisting of two sets and a binary relation between them, a relation which “join” elements in the two sets. In this aspect FCA and TJS resemble each other. But there are also differences. In TJS the joinings link elements in structures and the elements are themselves concepts. However, because of the formal similarity between FCA and TJS, methods and results from FCA can hopefully be applied in the development of TJS. The general character of FCA is emphasized by Davey and Priestley as follows:

> The framework within which we are working – a pair of sets, G, M, and a binary relation I linking them – is extremely general, and encompasses contexts which might not at first sight be viewed in terms of an object-attribute correspondence. Consider, for example, a computer program modelled by an input-output relation R between a finite set of initial states X and a finite set of final states Y with xRy if and only if the program when started in state x can terminate in state y. Then (X, Y, R) is the context for what is known as a (non-deterministic) transition system. Here A' (for $A \subseteq X$) is to be interpreted as the set of final states in which the program can terminate when started from any one of the states in A. (Davey & Priestly, 2002, p. 67.)

Formal concepts (or just concepts) in FCA are concepts of a special kind. They have intensions and extension as concepts usually have but of a special kind and are called intent and extent. The extent of a concept consists of objects that satisfy certain attributes, and these attributes constitute the intent of the concept. The extent and the intent of a concept constitute two different sets and an object is joined to an attribute if the object satisfies the attribute. The intended interpretation of a formal concept is a subset of what in TJS is considered as concepts, but is not the kind of concepts that TJS were primarily intended to capture.

We shall use FCA as a tool for the study of TJS. To avoid confusions with the notations in TJS some adjustments of the terminology and formalism used in FCA are necessary. The terms ‘objects’ and ‘attributes’ are not appropriate in TJS and will not be used here. (For the use of ‘attribute’ in this paper, see Section 4.) Even the notion ‘formal concept’ is problematic. As a preliminary solution to this problem the notion ‘conception’ will be used for ‘formal concept’. The notions ‘context’ and ‘conception’ will here be used as follows.

Suppose that $\mathcal{A}_1 = \langle A_1, \precsim_1 \rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2 \rangle$ are quasi-orderings and that $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is a protojoining-system. Then $\langle A_1, A_2, J \rangle$ is a *context* (a context based on quasi-orderings). Let $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$. Then define

$$C_1^{\vartriangle} = \{c_2 \in A_2 \mid (\forall c_1 \in C_1) \langle c_1, c_2 \rangle \in J\}$$

$$C_2^{\triangledown} = \{c_1 \in A_1 \mid (\forall c_2 \in C_2) \langle c_1, c_2 \rangle \in J\}.$$

Note that

$$\left(C_1^{\vartriangle}\right)^{\triangledown} = \left\{c_1 \in A_1 \mid \left(\forall c_2 \in C_1^{\vartriangle}\right)\langle c_1, c_2\rangle \in J\right\}$$

and, furthermore,

$$\left(\left(C_1^{\vartriangle}\right)^{\triangledown}\right)^{\vartriangle} = C_1^{\vartriangle}.$$

We often denote $\left(C_1^{\vartriangle}\right)^{\triangledown}$ with $C_1^{\vartriangle\triangledown}$ and $\left(\left(C_1^{\vartriangle}\right)^{\triangledown}\right)^{\vartriangle}$ with $C_1^{\vartriangle\triangledown\vartriangle}$. If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a protojoining-system (*pJs*) and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ such that $C_1^{\vartriangle} = C_2$ and $C_2^{\triangledown} = C_1$, then (C_1, C_2) is a *conception* in the context $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. The set of conceptions in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is denoted $\mathfrak{B}(\mathcal{A}_1, \mathcal{A}_2, J)$ or, when there is not risk of confusion, just $\mathfrak{B}(J)$. If (C_1, C_2) is a conception in the *pJs* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ then C_2 is an up-set with respect to $\precsim_2$ and C_1 is a down-set with respect to $\precsim_1$.[10] The set of conceptions in a *pJs* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ has an interesting structure. Let us define an ordering relation on conceptions as follows. If $\langle C_1, C_2\rangle, \langle D_1, D_2\rangle \in \mathfrak{B}(\mathcal{A}_1, \mathcal{A}_2, J)$ then

$$\langle C_1, C_2\rangle \mathrel{\widehat{\subseteq}} \langle D_1, D_2\rangle \Longleftrightarrow C_1 \subseteq D_1 \;\&\; C_2 \supseteq D_2.$$

The partial ordering $\left\langle \mathfrak{B}(\mathcal{A}_1, \mathcal{A}_2, J), \widehat{\subseteq}\right\rangle$ is a complete lattice. (For a proof see for example Davey & Priestley (2002) p. 69.)

3.2 Three theorems on conceptions and minimal joinings

In this section, the following abbreviations will be used: ‘*Js*’ for joining-system, ‘*Js(i)*’ for the i:th condition in the definition of a joining-system and ‘*pJs*’ for protojoining-system.

The three theorems in this subsection say roughly that in a *Js* a conception corresponds to a minimal joining. It is an interesting fact especially because minimal joinings are closely connected to intermediate concepts.

Suppose that $\langle A_1, A_2, J\rangle$ is a *pJs* and that $a_1 \in A_1$ and $a_2 \in A_2$. Then

$$\precsim_1^{-1}[a_1] = \{x_1 \in A_1 \mid x_1 \precsim_1 a_1\}$$

$$\precsim_2[a_2] = \{x_2 \in A_2 \mid a_2 \precsim_2 x_2\}.$$

(See the subsection on correspondences in subsection 1.3.2.)

Note that

$$\precsim_1^{-1}[a_1]^{\vartriangle} = \left\{b_2 \in A_2 \mid \left(\forall b_1 \in \precsim_1^{-1}[a_1]\right)\langle b_1, b_2\rangle \in J\right\}$$

$$\precsim_2[a_2]^{\triangledown} = \{b_1 \in A_1 \mid (\forall b_2 \in \precsim_2[a_2])\langle b_1, b_2\rangle \in J\}.$$

In this subsection we denote for simplicity the narrowness-relation of J with $\trianglelefteq$ instead of $\trianglelefteq_{1,2}$ and the lowerness-relation of J with $\precsim^*$ instead of $\precsim_{1,2}^*$.

[10] For definitions of ‘up-set’ and ‘down-set’ see for example Davey & Priestley (2002) p. 20. Cf. Lindahl & Odelstad (2013) p. 570 Definition 3.10.

Theorem 24 *Suppose that* $\langle A_1, A_2, J\rangle$ *is a* pJs *and* $\langle a_1, a_2\rangle \in \min_{\trianglelefteq} J$*, i.e.* $\langle a_1, a_2\rangle$ *is a minimal element in* J *with respect to* $\trianglelefteq$*. Then*

$$(1) \quad \precsim_1^{-1}[a_1]^{\vartriangle} = \precsim_2 [a_2]$$

$$(2) \quad \precsim_2 [a_2]^{\triangledown} = \precsim_1^{-1}[a_1]$$

i.e. $\left(\precsim_1^{-1}[a_1], \precsim_2 [a_2]\right)$ *is a conception in the context* $\langle A_1, A_2, J\rangle$*.*

Proof. Proof of (1). Since $\langle a_1, a_2\rangle \in \min_{\trianglelefteq} J$ and thus $\langle a_1, a_2\rangle \in J$ it follows that for all $b_1 \in \precsim_1^{-1}[a_1]$ it holds that $\langle b_1, a_2\rangle \in J$. Suppose that $b_2 \in \precsim_1^{-1}[a_1]^{\vartriangle}$. Then $\langle a_1, b_2\rangle \in J$. But since $\langle a_1, a_2\rangle \in \min_{\trianglelefteq} J$ it follows that $a_2 \precsim_2 b_2$. Since for all $b_1 \in \precsim_1^{-1}[a_1]$ it holds that $\langle b_1, a_2\rangle \in J$, it follows for all $b_1 \in \precsim_1^{-1}[a_1]$ that $b_1 \in \precsim_2 [a_2]^{\triangledown}$. Hence, $\precsim_1^{-1}[a_1]^{\vartriangle} \subseteq \precsim_2 [a_2]$.

Suppose now that $b_2 \in \precsim_2 [a_2]$. Then for all $b_1 \in \precsim_1^{-1}[a_1]$ it follows that $b_1 \precsim_1 a_1$, $a_1 J a_2$, $a_2 \precsim_2 b_2$, which implies that $\langle b_1, b_2\rangle \in J$. Hence $\precsim_2 [a_2] \subseteq \precsim_1^{-1}[a_1]^{\vartriangle}$. Together with $\precsim_1^{-1}[a_1]^{\vartriangle} \subseteq \precsim_2 [a_2]$ this implies (1).

Proof of (2). Suppose that $b_1 \in \precsim_2 [a_2]^{\triangledown}$. Then for all $b_2 \in \precsim_2 [a_2]$, $\langle b_1, b_2\rangle \in J$ and hence $\langle b_1, a_2\rangle \in J$. Since $\langle a_1, a_2\rangle \in \min_{\trianglelefteq} J$ it follows that $b_1 \precsim_1 a_1$ and thus $b_1 \in \precsim_1^{-1}[a_1]$. Hence, $\precsim_2 [a_2]^{\triangledown} \subseteq \precsim_1^{-1}[a_1]$.

Suppose that $b_1 \in \precsim_1^{-1}[a_1]$. Then $\langle b_1, a_2\rangle \in J$ and thus $\langle b_1, b_2\rangle \in J$ for all $b_2 \in \precsim_2 [a_2]$. Hence, $b_1 \in \precsim_2 [a_2]^{\triangledown}$. We have thus shown that $\precsim_1^{-1}[a_1] \subseteq \precsim_2 [a_2]^{\triangledown}$. Together with $\precsim_2 [a_2]^{\triangledown} \subseteq \precsim_1^{-1}[a_1]$ follows (2). ■

Theorem 25 *Suppose* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a* Js *such that* $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ *and* $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ *are complete quasi-lattices. If* $(C_1, C_2) \in \mathfrak{B}(\mathcal{A}_1, \mathcal{A}_2, J)$ *then there is* $b_1 \in A_1$ *and* $b_2 \in A_2$ *such that* $\langle b_1, b_2\rangle \in \min_{\trianglelefteq} J$ *and*

$$C_1 = \precsim_1^{-1}[b_1]$$

$$C_2 = \precsim_2 [b_2].$$

In the proof of the theorem we use the following lemma.

Lemma 26 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a* Js *such that* $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ *and* $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ *are quasi-orderings. Let* $C_1 \subseteq A_1$ *such that* $C_1^{\vartriangle} \neq \varnothing$*. Then the following holds:*

(1) $b_1 \in C_1^{\vartriangle\triangledown}$ *for all* $b_1 \in \mathrm{lub}_{\precsim_1} C_1$

(2) $b_2 \in C_1^{\vartriangle}$ *for all* $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$.

Proof. Proof of (1). Suppose that $a_2 \in C_1^{\vartriangle}$. Then for all $c_1 \in C_1$, $\langle c_1, a_2\rangle \in J$. From *Js(2)* it follows that $\langle b_1, a_2\rangle \in J$ for all $b_1 \in \mathrm{lub}_{\precsim_1} C_1$. Since a_2 is an arbitrary element in $C_1^{\vartriangle}$ it follows that for all $c_2 \in C_1^{\vartriangle}$ that $\langle b_1, c_2\rangle \in J$ for all $b_1 \in \mathrm{lub}_{\precsim_1} C_1$. Hence, $b_1 \in C_1^{\vartriangle\triangledown}$ for all $b_1 \in \mathrm{lub}_{\precsim_1} C_1$.

Proof of (2). Suppose that $a_1 \in C_1^{\vartriangle\triangledown}$. Then for all $c_2 \in C_1^{\vartriangle}$, $\langle a_1, c_2\rangle \in J$. From *Js(3)* it follows that $\langle a_1, b_2\rangle \in J$ for all $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$. Since a_1 is an arbitrary element in $C_1^{\vartriangle\triangledown}$ it follows that for all $c_1 \in C_1^{\vartriangle\triangledown}$ that $\langle c_1, b_2\rangle \in J$ for all $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$. Hence, $b_2 \in C_1^{\vartriangle\triangledown\vartriangle}$ and since $C_1^{\vartriangle\triangledown\vartriangle} = C_1^{\vartriangle}$ it follows that $b_2 \in C_1^{\vartriangle}$ for all $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$. ■

The proof of the theorem follows below.

Proof. Since (C_1, C_2) is a conception it follows that $C_1^{\vartriangle} = C_2$ and $C_2^{\triangledown} = C_1$. Hence, $C_1^{\vartriangle\triangledown} = C_2^{\triangledown} = C_1$. Since $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ are complete quasi-lattices then $\mathrm{lub}_{\precsim_1} C_1 \neq \varnothing$, i.e. $\mathrm{lub}_{\precsim_1} C_1^{\triangledown\vartriangle} \neq \varnothing$, and $\mathrm{glb}_{\precsim_2} C_1^{\triangledown} \neq \varnothing$.

Suppose that $b_1 \in \mathrm{lub}_{\precsim_1} C_1$ and $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$. Then, according to (1) in the lemma, $b_1 \in C_1^{\vartriangle\triangledown}$ and, according to (2), $b_2 \in C_1^{\vartriangle}$. Since $C_1^{\vartriangle\triangledown} = C_1$ it follows that $b_1 \in C_1$, which together with $b_2 \in C_1^{\vartriangle}$ implies that $b_1 J b_2$. Suppose now that $\langle d_1, d_2\rangle \in J$ such that $\langle d_1, d_2\rangle \trianglelefteq \langle b_1, b_2\rangle$. Hence, $b_1 \precsim_1 d_1$ and $d_2 \precsim_2 b_2$. Note that

$$d_1 J d_2 \;\&\; d_2 \precsim_2 b_2 \;\&\; b_2 \precsim_2 c_2$$

and, hence, $d_1 J c_2$ where c_2 is an arbitrary element in $C_1^{\vartriangle}$. Hence, $d_1 \in C_1^{\vartriangle\triangledown}$. Note that

$$c_1 \precsim_1 b_1 \;\&\; b_1 \precsim_1 d_1 \;\&\; d_1 J_2 d_2$$

where c_1 is an arbitrary element in C_1. Hence, $d_2 \in C_1^{\vartriangle}$.

Since $C_1^{\vartriangle\triangledown} = C_1$ and $d_1 \in C_1^{\vartriangle\triangledown}$ it follows that $d_1 \in C_1$. Since $b_1 \in \mathrm{lub}_{\precsim_1} C_1$ it follows that $d_1 \precsim_1 b_1$. From $d_2 \in C_1^{\vartriangle}$ and $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$ it follows that $b_2 \precsim_2 d_2$. Together with $b_1 \precsim_1 d_1$ and $d_2 \precsim_2 b_2$ it follows that $b_1 \sim_1 d_1$ and $b_2 \sim_2 d_2$. Hence $\langle b_1, b_2\rangle \in \min_{\trianglelefteq} J$. From the previous theorem follows that $\left(\precsim_1^{-1}[b_1], \precsim_2[b_2]\right)$ is a conception in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$.

We show now that

$$\text{(i)}\;\; C_1 = \precsim_1^{-1}[b_1]$$

$$\text{(ii)}\;\; C_2 = \precsim_2[b_2].$$

(i) Suppose that $a_1 \in C_1$. Since $b_1 \in \mathrm{lub}_{\precsim_1} C_1$ it follows that $a_1 \precsim_1 b_1$ and, hence, $a_1 \in \precsim_1^{-1}[b_1]$. This shows that $C_1 \subseteq \precsim_1^{-1}[b_1]$. Suppose that $a_1 \in \precsim_1^{-1}[b_1]$. Then $a_1 \precsim_1 b_1$ and since $b_1 J_2 b_2$ it follows that $a_1 J_2 b_2$, and since $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$ it follows for all $c_2 \in C_1^{\vartriangle}$ that $a_1 J_2 c_2$ and, hence, $a_1 \in C_1^{\vartriangle\triangledown}$ and since $C_1^{\vartriangle\triangledown} = C_1$ it follows that $a_1 \in C_1$. This shows that $\precsim_1^{-1}[b_1] \subseteq C_1$ and it follows that $C_1 = \precsim_1^{-1}[b_1]$.

(ii) Suppose that $a_2 \in C_2$. Since $b_2 \in \mathrm{glb}_{\precsim_2} C_1^{\vartriangle}$ and $C_1^{\vartriangle} = C_2$ it follows that $b_2 \precsim_2 a_2$ and, hence, $a_2 \in \precsim_2[b_2]$. This shows that $C_2 \subseteq \precsim_2[b_2]$. Suppose that $a_2 \in \precsim_2[b_2]$. Then $b_2 \precsim_2 a_2$ and since $b_1 J_2 b_2$ it follows that $b_1 J_2 a_2$, and from $b_1 \in \mathrm{lub}_{\precsim_1} C_1$ follows that $c_1 J_2 a_2$ for all $c_1 \in C_1$ and, hence, $a_2 \in C_1^{\vartriangle}$. Since $C_1^{\vartriangle} = C_2$ it follows that $a_2 \in C_2$, which shows that $\precsim_2[b_2] \subseteq C_2$ and it follows that $C_2 = \precsim_2[b_2]$. ■

The two theorems above in this subsection show that if $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ are complete quasi-lattices and $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system then there is a correspondence between the elements in $\min_{\trianglelefteq} J$ and the set of conceptions in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. The following theorem shows that this correspondence is in fact a homomorphism on $\left\langle \min_{\trianglelefteq} J, \precsim^*\right\rangle$ onto $\left\langle \mathfrak{B}(J), \widehat{\subseteq}\right\rangle$. As pointed out above $\left\langle \mathfrak{B}(J), \widehat{\subseteq}\right\rangle$ is a complete lattice. This fact together with the theorem below is thus related to Corollary 3.36 (p. 588) in Lindahl & Odelstad (2013). (Note that the relation $\precsim^*$ in the Corollary 3.36 is not exactly the same as the relation $\precsim^*$ in the theorem below.)

Theorem 27 *Suppose that $\mathcal{A}_1 = \langle A_1, \precsim_1\rangle$ and $\mathcal{A}_2 = \langle A_2, \precsim_2\rangle$ are complete quasi-lattices and $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a* Js. *Let*

$$\varphi : \min_{\trianglelefteq} J \to \mathfrak{B}(J)$$

such that

$$\varphi(\langle a_1, a_2\rangle) = \left(\precsim_1^{-1}[a_1], \precsim_2[a_2]\right).$$

Then φ *is a homomorphism on* $\left\langle \min_{\trianglelefteq} J, \precsim^* \right\rangle$ *onto* $\left\langle \mathfrak{B}(J), \widehat{\subseteq} \right\rangle$.

Proof. According to Theorem 24, $\left(\precsim_1^{-1}[a_1], \precsim_2[a_2]\right)$ is a conception in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$, i.e. φ assigns to every element in $\min_{\trianglelefteq} J$ a corresponding conception in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$.

(I) We first prove that φ is onto $\mathfrak{B}(J)$. Suppose that $\langle C_1, C_2\rangle \in \mathfrak{B}(J)$. Then, according to Theorem 25, there is $\langle c_1, c_2\rangle \in \min_{\trianglelefteq} J$ such that

$$C_1 = \precsim_1^{-1}[c_1]$$
$$C_2 = \precsim_2[c_2].$$

Hence,

$$\varphi(\langle c_1, c_2\rangle) = \left(\precsim_1^{-1}[c_1], \precsim_2[c_2]\right) = (C_1, C_2).$$

(II) We now prove that

$$\langle a_1, a_2\rangle \precsim^* \langle b_1, b_2\rangle \;\Leftrightarrow\; \varphi(\langle a_1, a_2\rangle) \;\widehat{\subseteq}\; \varphi(\langle b_1, b_2\rangle).$$

Suppose that $\langle a_1, a_2\rangle \precsim^* \langle b_1, b_2\rangle$. Then $a_1 \precsim_1 b_1$ and $a_2 \precsim_2 b_2$ and, hence, $a_1 \in \precsim_1^{-1}[b_1]$ and $b_2 \in \precsim_2[a_2]$. If $x_1 \in \precsim_1^{-1}[a_1]$ then $x_1 \precsim_1 a_1$ which implies that $x_1 \precsim_1 b_1$ and, further, $x_1 \in \precsim_1^{-1}[b_1]$. This shows that

$$\precsim_1^{-1}[a_1] \subseteq \precsim_1^{-1}[b_1].$$

If $x_2 \in \precsim_2[b_2]$ then $b_2 \precsim_2 x_2$ which implies that $a_2 \precsim_2 x_2$ and, further, $x_2 \in \precsim_2[a_2]$. This shows that

$$\precsim_2[b_2] \subseteq \precsim_2[a_2].$$

We have thus proved that

$$\precsim_1^{-1}[a_1] \subseteq \precsim_1^{-1}[b_1] \quad \& \quad \precsim_2[a_2] \supseteq \precsim_2[b_2]$$

which is equivalent to

$$\left(\precsim_1^{-1}[a_1], \precsim_2[a_2]\right) \;\widehat{\subseteq}\; \left(\precsim_1^{-1}[b_1], \precsim_2[b_2]\right)$$

i.e.

$$\varphi(\langle a_1, a_2\rangle) \;\widehat{\subseteq}\; \varphi(\langle b_1, b_2\rangle).$$

This shows that

$$\langle a_1, a_2\rangle \precsim^* \langle b_1, b_2\rangle \;\Rightarrow\; \varphi(\langle a_1, a_2\rangle) \;\widehat{\subseteq}\; \varphi(\langle b_1, b_2\rangle).$$

Suppose now that $\varphi(\langle a_1, a_2\rangle) \;\widehat{\subseteq}\; \varphi(\langle b_1, b_2\rangle)$ from which follows that

$$\precsim_1^{-1}[a_1] \subseteq \precsim_1^{-1}[b_1] \quad \& \quad \precsim_2[a_2] \supseteq \precsim_2[b_2].$$

This implies that $a_1 \in \precsim_1^{-1}[b_1]$ and $b_2 \in \precsim_2[a_2]$ and, hence, $a_1 \precsim_1 b_1$ and $a_2 \precsim_2 b_2$, which shows that

$$\varphi(\langle a_1, a_2\rangle) \;\widehat{\subseteq}\; \varphi(\langle b_1, b_2\rangle) \;\Rightarrow\; \langle a_1, a_2\rangle \precsim^* \langle b_1, b_2\rangle.$$

■

4 Joining systems of aspects

4.1 Aspects

The *cis* model of TJS was developed primarily as a framework for representing legal systems and other normative systems based on implications between conditions. However, as has been pointed out in Section 1, TJS can be used as a framework for other kinds of *msic*-systems, too. In this section we will focus on *aspects*, in many disciplines called attributes but here ‘aspect’ will be preferred. Well-known examples of aspects are area, temperature, age, loudness and archeological value. Some kinds of aspects have special names, primarily in certain contexts, for example quantity, quality, criterion, feature, characteristic, property, indicator, dimension or magnitude. Quantitative aspects, i.e. quantities, are usually measurable and such aspects are not seldom confused with a measure of it (utility is one example). It is a common view of aspects that they can, in some way or another, from a formal point of view be represented as relational structures. We will return to the formal representation of aspects below.

Conditions can be of different sorts and the same holds for aspects. Like conditions, aspects can, among other things, be descriptive or normative (i.e. evaluating) or something in between, which means that they are intermediaries between descriptive and normative conceptual systems. There are aspects which are intermediate between conceptual systems of other sorts, too, but we will in the informal part of this section focus on intermediaries between descriptive and normative aspects.

In the *cis* model a conceptual system consists of conditions and the relation of implication between them. In a conceptual system of aspects there is an implicative relation from its grounds to its consequences, but this relation is not sentential implication. (See further Section 4.4.2.)

The meaning of an intermediate aspect consists jointly of stating its descriptive grounds and its normative (evaluative) consequences. Which conceptual systems that intermediate aspects are in between, are often not immediately evident. In many contexts it can be more informative to regard them as concepts determined by grounds and consequences and initially leaving open the exact character of the top and the bottom conceptual systems. In such cases the aspects can be called ground-consequence concepts.[11]

Ground-consequence concepts are of course related to what is often called thick concepts. The philosophical discussion of such concepts, especially in ethics, is relevant for the understanding of ground-consequence concepts.[12] But this line of thought will not be pursued here. The concepts that are in focus in this section are not those

[11]Cf. Odelstad (2002), especially Chapter 12.

[12]Lacey (1996) p. 347 headword ‘Thick and thin concepts’:

> Terms used especially in recent ethics. Thick concepts are those which seem to combine a purely descriptive element with an element of evaluation or prescription, such as ‘cowardly’, ‘heroic’, ‘treacherous’, ‘loyal’, ‘brutal’, ‘lewd’, while thin terms embody only an evaluative or prescriptive element, such as ‘good’, ‘evil’, ‘ought’, ‘right’. It seems hard for someone who does not accept the relevant values or prescriptions to decide whether to call attributions of the thick concepts true or false. However, the correct analysis of the thick concepts is disputed.

usually discussed in moral philosophy but are instead of importance for planning as well as decision and policy making, i.e. so-called policy relevant concepts. Among such concepts normative indicators play an important role. In Sen (1977) the following examples are mentioned.

> *Normative indication:* Measurement of "national income", "inequality", "poverty", and other "indicators" defined with normative motivation incorporating interpersonal weighting in some easily tractable way. (Sen, 1977, p. 53.)

Other examples are gross domestic product, inflation, unemployment, gender equality, public interest, archeological value and accessibility to social services. In Páez et.al. (2012) the authors discuss measuring of accessibility in the transportation sector focusing on positive and normative implementations of various accessibility indicators. In the abstract the authors emphasize the following:

> Accessibility is a concept of continuing relevance in transportation research. A number of different measures of accessibility, defined as the potential to reach spatially dispersed opportunities, have been proposed in the literature, and used to address various substantive planning and policy questions. Our objective in this paper is to conduct a review of various commonly used measures of accessibility, with a particular view to clarifying their normative (i.e. prescriptive), as well as positive (i.e. descriptive) aspects. This is a distinction that has seldom been made in the literature and that helps to better understand the meaning of alternative ways to implement the concept of accessibility. (Páez et.al., 2012, p. 141.)

Accessibility in the transportation sector seems to be a nice example of a policy-relevant intermediate aspect and since accessibility is a quantity the question of its measurement is relevant.

The problem how an aspect ought to be measured can be interpreted in different ways. Suppose that the meaning of the aspect α is a joining of descriptive grounds and normative consequences. When you determine how such an aspect ought to be measured you take a normative stand. Grounds and consequences must match each other, which is a normative problem. The decision how to measure is a part of clarifying the meaning of the concept.

In many contexts, for example in multi-criteria decision analysis, there are concepts joining descriptive grounds and normative consequences which are ground open (and perhaps even consequence open).[13] They function as decision nodes in the step by step decision process where the decision is partially determined by the grounds and

[13]The notions 'ground open' and 'consequence open' are explained, exemplified and discussed in Lindahl & Odelstad (2013) pp. 557–559 and 617–620. A short remark on the notions based on Odelstad (2009) pp. 15–16 follows below.

The concept 'work of equal value', which is an essential concept in the Swedish Equal Opportunities Act, is an intermediary with one face looking at the nature of and requirements for the work and the other face looking at efforts to promote equality in working life, especially equal pay for equal work. The law does not supply us with a complete set of introduction rules for the concept. Instead it mentions some criteria that equality of value depends on, viz. knowledge and skills, responsibility and effort. The applicability of the

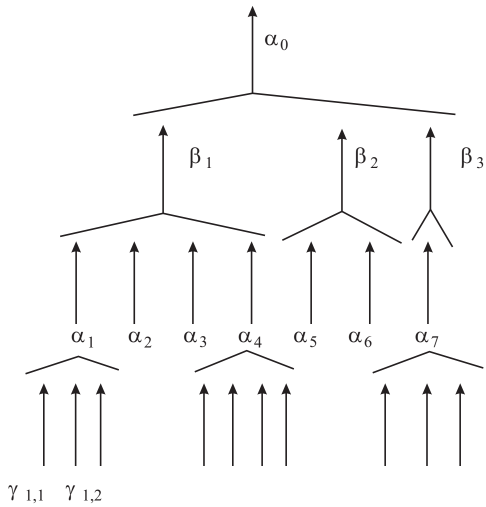

Figure 5: Aggregation tree with four strata.

consequences of the concept. One important part of the decision process in a multi-criteria decision problem is the aggregation of the different factors or components that influence the value of the outcomes. The result of the aggregation is the value of the outcomes all things considered. In Figure 5 a simple aggregation is pictured as an aggregation tree. The factors $\gamma_{i,j}$ at the bottom are descriptive aspects and the aggregate α_0 at the top states the value all things considered and has purely normative consequences (for example in terms of what ought to be chosen). The aspects α_i and β_j are intermediaries, where the β_j:s represent "higher, more normative" strata than the α_i:s.

In Lindahl & Odelstad (2008) p. 205 it is emphasized that the pattern of a comprehensive system of legal concepts is usually that of a network of structures of intermediate concepts and this is illustrated as the middle part of Figure 6. (Note that the consequences of one intermediate concept can be the grounds of another.)

The similarity between the network of structures of intermediate legal concepts and the strata of decision analytic intermediate aspects is from a structural perspective obvious, which is illustrated in Figure 7. Note that the input of facts refer to "factual", i.e. descriptive, aspects and not to extensions of descriptive facts. (Extensions of aspects will be discussed in subsection 4.4.)

We end this section with a fictitious but not unrealistic example. Assume that a

concept work of equal value in a certain case must often be based on judgments of what holds in the actual case. And even if the law does not state detailed rules for these judgments it gives guidelines, for example in terms of what are possible inputs in such judgments or what factors or circumstances must be taken into account. The grounds of the concept 'work of equal' value is thus only partially determined by the law in the form of introduction rules. The application of the concept in special cases deserves interpretative decisions based on the role and function of the concept in the law. We call such intermediaries *ground-open*. Concepts such that the consequences are only partially determined by elimination rules are called *consequence-open*.

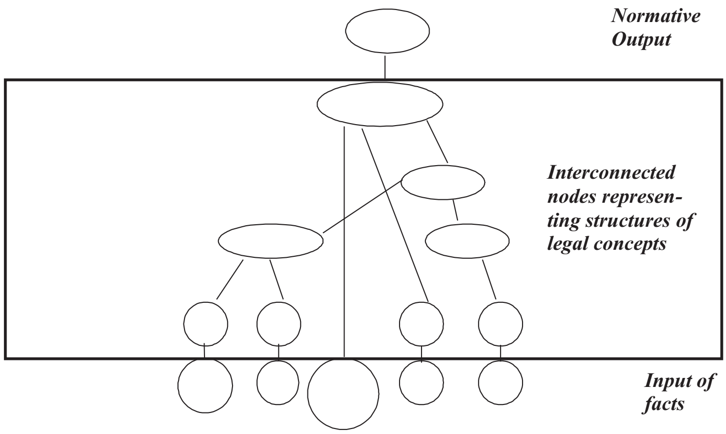

Figure 6: A legal system as a network of strata.

county council has adopted a public transport policy. Part of the policy concerns bus traffic, including stipulations that the punctuality of buses should be gradually increased by at least 5% per year, which is an objective expressed in quantitative terms. In order to evaluate the effects of the policy, one must examine whether the objectives are met. In that case, the punctuality of the buses must be measured. Measuring how much a bus is delayed at a certain occasion is quite simple. But measuring how much all buses used by a bus company are delayed during one month seems to be more complicated. Assume that bus company *A* has many buses that are a bit late while bus company *B* has few buses that are late, but instead they are very late. How do you compare these two outcomes to each other, which is the worst? Does it further matter if late buses have many or few passengers and if the delays occur in rush or in low traffic?

The punctuality of a class of buses is, as emphasized above, a multi-dimensional concept. Measurement with regard to this concept is not straightforward. Should you try to find an aspect of punctuality that is easy to measure and see it as an operationalization of punctuality? Or should you measure with regard to as many dimensions of punctuality as possible and aggregate them? Or should you choose some particularly important dimensions and aggregate them? Anyway, the choice will involve valuations. Punctuality of buses is not a directly measurable concept, not even a purely descriptive concept. On the other hand, the concept punctuality has a descriptive ground consisting of a number of different aspects, but to determine punctuality, one has to evaluate the ground with regard to the specific character of the normative consequences. Bus

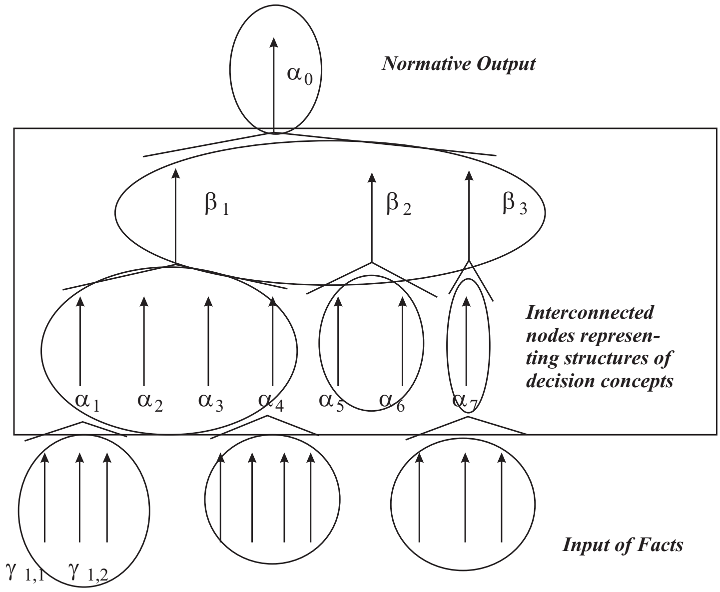

Figure 7: A multi-criteria decision system as a network of strata.

punctuality cannot be measured in strict terms but must be judged or evaluated based on measurements of the descriptive dimensions of the concept. The punctuality of the bus company depends on facts, but *how* involves valuations. Hence, the punctuality of buses is an intermediary that has descriptive grounds and normative consequences.

4.2 Relations and aspects as functions

Aspects are constituted by relations and operations in the sense that an aspect is a structure consisting of relations and operations. An operation can be regarded as a relation of a special kind, for example a binary operation can formally be understood as a ternary relation. In many situations this is not a wise procedure but, anyhow, in this paper I will temporarily accept this simplification and regard aspects as constituted by relations only.

Suppose that α is an aspect constituted by the relations $R_1, ..., R_n$. Then α is a structure with $R_1, ..., R_n$ as components. But which is the domain of the structure? To specify one special domain for the aspect and the relations is in many contexts awkward. For example, it is reasonable to think that a person's preference relation always is restricted to a special set of alternatives. Hence, that R_i, the individual i's preference relation on set A of alternatives, is ρ_i is best represented as $\langle A, \rho_i \rangle$, in other words

$$R_i(A) = \langle A, \rho_i \rangle .$$

From a formal standpoint this means that R_i is a function that takes sets of alternatives as arguments and a relation (or a relational structure) as values. Suppose now that this holds for all i, $1 \leq i \leq n$ (i can be individuals in a group that will make a decision), then

$$\alpha(A) = \langle A, R_1(A), ..., R_n(A) \rangle = \langle A, \rho_1, ..., \rho_n \rangle .$$

ρ_i as well as $\langle A, \rho_i \rangle$ can be regarded as the extension of R_i on A, and analogously, $\alpha(A)$ the extension of α on A. The domain of R_i and α as functions is a family of sets $\mathcal{D}$. If α is constituted by $R_1, ..., R_n$ then α can be represented as $\langle R_1, ..., R_n \rangle$ which is a function that takes elements in $\mathcal{D}$ as values. Hence,

$$\alpha = \langle R_1, ..., R_n \rangle .$$

Relations regarded as functions in the way just described will be called *relationals*.

Note that conditions are relations but conditions in legal systems are often of another type than the ordinary relations in aspects. However, legal conditions can of course be represented as relationals, see subsection 4.4.3. Furthermore, note that an aspect can be constituted by different but equivalent compositions of relationals.

In Odelstad (1992), a theory of relations and aspects as functions, primarily as base for a study of dependence and independence in systems of aspects, is presented. Some central notions and results will be presented in the next subsection. For an elaborate presentation of the theory, see Odelstad (1992).

This subsection ends with a short historical and philosophical remark on relationals. (For further details, see Odelstad, 1992, especially subsection 3.5 pp. 93–95.) One of the ideas behind the notion of a relational is that it is meaningful to talk about the extension of a relation over a set. This idea seems to be an old one which in different contexts has taken different forms. In *Meaning and necessity* Carnap suggested that certain intensional entities be identified with functions that take possible state of affairs as arguments. The value of such a function is the extension of the intensional entity at that state of affair. Carnap's idea has been developed further by Kanger, Kripke, Kaplan and Montague among others. Relationals resemble especially Montague's predicates in that they are functions with extensions as values. (See Montague, 1974, p. 152.) There are thus some formal similarities between relationals and some notions in theories of modal logic. Most striking is perhaps the similarity with Kanger's version of the semantics for modal logic since functions representing intensional objects there have domains as arguments. But the domains in the range of definition of a relational are not intended to represent possible worlds; even in the actual world, a relational can have different extensions depending on what domain one considers. The idea of intensional entities as functions which have extensions as values was first used in formal semantics, while the notion of a relational was intended to be used within measurement and decision theory. In this paper relationals is a part of the framework for the aspect model of TJS.

4.3 Relationals: some definitions and results

4.3.1 Basic terminology for relationals

Definition 28 *P is a ν-ary* one-component relational *with range of definition $\mathfrak{D}$ if for all $A \in \mathfrak{D}$*

$$P(A) = \langle A, \rho \rangle$$

where ρ is a ν-ary relation on A.

Definition 29 *P is a* relational of type $\langle \nu_1, ..., \nu_k \rangle$ *with range of definition $\mathfrak{D}$ if for all $A \in \mathfrak{D}$*

$$P(A) = \langle A, \rho_1, ..., \rho_k \rangle$$

where ρ_i is an ν_i-ary relation on A. P is a ν-ary one-component relational *if P is a relational of type $\langle \nu \rangle$. The range of definition of a relational P is a family of sets and is generally denoted by D_P, i.e. $\mathrm{D}_P = \mathfrak{D}$.*

If P is a relational of type $\langle \nu \rangle$ then $P(A) = \langle A, \rho \rangle$ where ρ is a ν-ary relation on A. For simplicity, instead of '$\rho(x_1, ..., x_\nu)$' we often use one of the following notations when there is no risk of ambiguity:

$$\langle x_1, ..., x_\nu \rangle \in P(A)$$
$$P(A; x_1, ..., x_\nu)$$
$$P(x_1, ..., x_\nu); A.$$

If $\succsim$ is a binary ordering relation then the following notations are used synonymously:

$$\langle x, y \rangle \in \succsim (A)$$
$$x \succsim y; A.$$

If $P(A) = \langle A, \rho \rangle$ then we say that $\langle A, \rho \rangle$ is the *graph* of P on A and that ρ is the *proper extension* of P on A. The term *extension* will be used for graph or proper extension in situations where it is clear from the context what is meant. The extension of P on A is thus $\langle A, \rho \rangle$ or ρ depending on the context.

Definition 30 *The* extension class *of the relational P, denoted E_P, is the set*

$$\{P(A) | A \in \mathrm{D}_P\}.$$

The characteristic class *of the relational P, denoted C_P, is E_P closed under isomorphisms, i.e.*

$$\mathrm{C}_P = \{\mathcal{X} \mid \exists A \in \mathrm{D}_P : \mathrm{I}(P(A), \mathcal{X}) \neq \varnothing\}.$$

Definition 31 *If for all i, $1 \leq i \leq k$, P_i is one-component relational of type $\langle \nu_i \rangle$ with the range of definition $\mathfrak{D}$, then the* concatenation *of $P_1, P_2, ..., P_k$ is the relational P of type $\langle \nu_1, ..., \nu_k \rangle$ such that for all $A \in \mathfrak{D}$*

$$P(A) = \langle A, \rho_1, ..., \rho_k \rangle$$

where

$$P_i(A) = \langle A, \rho_i \rangle .$$

The concatenation of $P_1, P_2, ..., P_k$ *is denoted* $(P_1, P_2, ..., P_k)$.

Note that

$$(P_1 P_2 ... P_k)(A) = \langle A, P_1(A), P_2(A), ..., P_k(A) \rangle$$

where $P_i(A) = \rho_i$, i.e. $P_i(A)$ is here the proper extension of P_i on A. The concatenation of P and Q are sometimes written PQ instead of (PQ).

If the aspect α is constituted by $\langle P_1, ..., P_n \rangle$ then we can regard α as constituted by one relational, viz. $(P_1 ... P_n)$, the concatenation of $P_1, ..., P_n$.

4.3.2 Subordination, superiority and rank

Definition 32 *Suppose that P and Q are relationals with the same range of definition* $\mathcal{D}$*. Q is* subordinate *to P, denoted by* $Q \Downarrow P$*, if for all* $A, B \in \mathcal{D}$

$$\mathrm{I}(Q(A), Q(B)) \supseteq \mathrm{I}(P(A), P(B)) .$$

If Q is subordinate *to P then P is* superior *to Q denoted by* $P \Uparrow Q$*. If Q is* subordinate *to P and P is* subordinate *to Q then P and Q are said to* be on a par*, which is denoted* $P \Updownarrow Q$*.*

The notion of subordination is closely related to definability. In first order logic a distinction between explicit and implicit definability is often made (see for example Chang & Keisler, 2012, p. 90) and the equivalence of the two notions in first-order logic is the celebrated Beth's theorem on definability. In the theory of relationals a formal language is not used and, hence, 'explicit definability' is not applicable here. But 'implicit definability' is, as a model-theoretic notion of definability, meaningful in the theory of relationals and is equivalent to 'subordination'. This is made clear by Theorem 37 in subsection 4.3.4. (This is a simplified presentation of definability in the context of relationals, for a more detailed presentation see Odelstad, 1992, especially subsection 4.2 pp. 103–105. Beth's theorem and Padoa's method are useful tools for a more profound study of subordination and transitions than what is aimed at in this paper.)

If an aspect α is constituted by the relational P and $P \Updownarrow Q$ then α can from a formal point of view be constituted by Q as well.

The following proposition is easily verified.

Proposition 33 *Let* $\mathfrak{R}$ *be a set of relationals with* $\mathcal{D}$ *as range of definition. Then* $\langle \mathfrak{R}, \Uparrow \rangle$ *is a quasi-ordering.*

Note that $P \Uparrow Q$ iff for all $A, B \in \mathcal{D}$

$$\mathrm{I}(P(A), P(B)) \subseteq \mathrm{I}(Q(A), Q(B))$$

and $P \Updownarrow Q$ iff for all $A, B \in \mathcal{D}$

$$\mathrm{I}(P(A), P(B)) = \mathrm{I}(Q(A), Q(B)) .$$

We introduce the following abbreviations:

$$\mathrm{I}_P(A,B) = \mathrm{I}(P(A), P(B)),$$

$$\mathrm{I}_P(A,A) = \mathrm{I}_P(A),$$

$$\mathrm{Bi}(\mathcal{D}\times\mathcal{D}) = \{\mathrm{Bi}(A,B) \mid A,B \in D\}.$$

Note that

$$\mathrm{I}_P(A,B) \subseteq \mathrm{Bi}(A,B).$$

Let us call I_P the *rank* of P.

We infer componentwise versions of $\subseteq$ and $\cap$ and denote them $\dot{\subseteq}$ and $\dot{\cap}$: Suppose that F and G are functions from $\mathcal{D}\times\mathcal{D}$ into $\mathrm{Bi}(\mathcal{D}\times\mathcal{D})$ such that $F(A,B), G(A,B) \subseteq \mathrm{Bi}(A,B)$. Then for all $A,B \in \mathcal{D}$

$$\left(F \dot{\cap} G\right)(A,B) = F(A,B) \cap G(A,B)$$

$$F \dot{\subseteq} G \text{ iff } \forall A,B \in \mathcal{D} : F(A,B) \subseteq G(A,B).$$

We define $\doteq$ as follows:

$$F \doteq G \text{ iff } F \dot{\subseteq} G \;\&\; G \dot{\subseteq} F.$$

Hence,

$$\left(\mathrm{I}_P \dot{\cap} \mathrm{I}_Q\right)(A,B) = \mathrm{I}_P(A,B) \cap \mathrm{I}_Q(A,B)$$

$$\mathrm{I}_P \dot{\subseteq} \mathrm{I}_Q \text{ iff } \forall A,B \in \mathcal{D} : \mathrm{I}_P(A,B) \subseteq \mathrm{I}_Q(A,B).$$

Note that $R \Uparrow S$ iff $\mathrm{I}_R \dot{\subseteq} \mathrm{I}_S$. And $R \Updownarrow S$ iff $\mathrm{I}_R \doteq \mathrm{I}_S$. Hence, the quasi-ordering $\Uparrow$ on a set of relationals corresponds to the partial ordering $\dot{\subseteq}$ on the ranks of the relationals.

There are other interesting quasi-orderings on relationals than subordination and two such quasi-orderings will be mentioned here but not further studied in this paper.

Definition 34 *Suppose that P and Q are relationals with the same range of definition $\mathcal{D}$. Q is automorphically* subordinate *to P, denoted by $Q \Downarrow_a P$, if for all $A \in \mathcal{D}$*

$$\mathrm{I}(Q(A)) \supseteq \mathrm{I}(P(A)).$$

If $Q \Downarrow_a P$ and $P \Downarrow_a Q$ then Q and P are said to be automorphically on a par, which is denoted $P \Updownarrow_a Q$.

Let $\mathfrak{C}(\mathcal{X})$ be the set of congruence relations on $\mathcal{X}$.

Definition 35 *Suppose that P and Q are relationals with the same range of definition $\mathcal{D}$. Q* conforms to *P, denoted by $P \uparrow Q$ if for all $A \in \mathcal{D}$*

$$\mathfrak{C}(S(A)) \supseteq \mathfrak{C}(R(A)).$$

If Q conforms to P and P conforms to Q then P and Q are said to be equiform.

4.3.3 Transitions

In the rest of this paper we suppose that all relationals that are the subject of study have the range of definition $\mathcal{D}$.

Definition 36 *The transition from P to Q, denoted P^Q, is the correspondence from* C_P *to* C_Q *defined by*

$$P^Q = \{\langle \mathcal{X}, \mathcal{Y} \rangle \in \mathrm{C}_P \times \mathrm{C}_Q \mid X = Y \,\&\, \exists Z \in \mathcal{D} : \mathrm{I}(\mathcal{X}, P(Z)) \cap \mathrm{I}(\mathcal{Y}, Q(Z)) \neq \varnothing\}.$$

The definition can also be written

$$P^Q = \left\{ \begin{array}{c} \langle \mathcal{X}, \mathcal{Y} \rangle \in \mathrm{C}_P \times \mathrm{C}_Q \mid X = Y \,\&\, \exists Z \in \mathcal{D} : \exists \varphi \in \mathrm{Bi}(X, Z) : \\ \mathcal{X} = \varphi^{-1}[P(Z)] \text{ and } \mathcal{Y} = \varphi^{-1}[Q(Z)] \end{array} \right\}$$

In the sequel, $\langle \mathcal{X}, \mathcal{Y} \rangle \in P^Q$ is often written $\mathcal{X} P^Q \mathcal{Y}$. Note that $P(A)\, P^Q Q(A)$.

The transition P^Q is a correspondence from C_P to C_Q, which we also express by saying that $\left\langle \mathrm{C}_P, \mathrm{C}_Q, P^Q \right\rangle$ is a correspondence. P^Q is thus a correspondence with C_P as domain and C_Q as image. When there is no risk of ambiguity we omit the references to domain and image and say that P^Q is a correspondence.

Note that since P^Q is a correspondence from C_P to C_Q, if $\mathcal{X} \in \mathrm{C}_P$ then

$$P^Q[\mathcal{X}] = \left\{ \mathcal{Y} \in \mathrm{C}_Q \mid \mathcal{X} P^Q \mathcal{Y} \right\}.$$

Note further that if $P(A) = \mathcal{A}$ then $\mathcal{A} P^Q Q(A)$, i.e. $Q(A) \in P^Q[\mathcal{A}]$. Hence, if $P(A) = \mathcal{A}$ then the possible extensions of Q over A are elements in $P^Q[\mathcal{A}]$. And furthermore, the elements in $P^Q[\mathcal{A}]$ are the possible values for $Q(A)$ given that $P(A) = \mathcal{A}$ and no other information is accessible.

A transition P^Q is *closed under isomorphisms* in the following sense: If $\mathcal{X} P^Q \mathcal{Y}$, $\varphi \in \mathrm{I}(\mathcal{X}, \mathcal{X}')$, $\varphi \in \mathrm{I}(\mathcal{Y}, \mathcal{Y}')$ then $\mathcal{X}' P^Q \mathcal{Y}'$. Note that this condition can also be stated as If $\mathcal{X} P^Q \mathcal{Y}$, and $\varphi \in \mathrm{I}(\mathcal{X}, \mathcal{X}')$ then $\mathcal{X}' P^Q \varphi[\mathcal{Y}]$. The notion 'closed under isomorphism' can be extended to a correspondence Γ from a set $\mathcal{K}_1$ of structures of the same type τ_1 (closed under isomorphisms) to a set $\mathcal{K}_2$ of structures of the same type τ_2 (closed under isomorphisms) in the following way: If $\mathcal{X} \Gamma \mathcal{Y}$, $\varphi \in \mathrm{I}(\mathcal{X}, \mathcal{X}')$, $\varphi \in \mathrm{I}(\mathcal{Y}, \mathcal{Y}')$ then $\mathcal{X}' \Gamma \mathcal{Y}'$. This condition can also be stated as follows: If $\mathcal{X} \Gamma \mathcal{Y}$ and $\varphi \in \mathrm{I}(\mathcal{X}, \mathcal{X}')$ then $\mathcal{X} \Gamma \varphi[\mathcal{Y}]$.

Note that if $P^Q = \Gamma$ then the following holds: (1) $\Gamma \subseteq \mathrm{C}_P \times \mathrm{C}_Q$, (2) $P(A)\, \Gamma\, Q(A)$, (3) $\mathcal{X} \Gamma \mathcal{Y}$ implies $X = Y$ and (4) Γ is closed under isomorphisms and (5) if $\mathcal{X} \Gamma \mathcal{Y}$ then there is $A \in \mathcal{D}$ and $\varphi \in \mathrm{I}(\mathcal{X}, P(A))$ such that $\varphi \in \mathrm{I}(\mathcal{Y}, Q(A))$. These five conditions on Γ are also sufficient for $P^Q = \Gamma$. If $P^Q = \Gamma$ then Γ together with P confines the set of possible values for Q given the value of P in the following sense $Q(A) \in (P|\Gamma)[A]$, and we say that Q is *partially determined* by P and Γ.

4.3.4 Transitions as functions

Note that P^Q is a function iff $P^Q[\mathcal{X}]$ contains exactly one element for all $\mathcal{X} \in \mathrm{C}_P$. Then P determines Q completely.

Theorem 37 *$P \Uparrow Q$ iff P^Q is a function on* C_P *onto* C_Q.

For a proof see Odelstad (1992) Theorem 4.1.1.

If a transition P^Q is a function then P^Q is *isomorphism-preserving* in the following sense: If $\mathcal{X}P^Q\mathcal{Y}$, $\mathcal{X}'P^Q\mathcal{Y}'$ and $\varphi \in \mathrm{I}(\mathcal{X},\mathcal{X}')$ then $\varphi \in \mathrm{I}(\mathcal{Y},\mathcal{Y}')$.

Note that P and Q are functions and the same holds for P^Q if $P \Uparrow Q$:

$$P : \mathcal{D} \to \mathrm{E}_P, \quad Q : \mathcal{D} \to \mathrm{E}_Q, \quad P^Q : \mathrm{C}_P \to \mathrm{C}_Q.$$

The two lemmas below show the interrelationship between these functions. First we observe the following. The notion of isomorphism-preservation can be extended to a function on a set $\mathcal{K}_1$ of structures of the same type τ_1 closed under isomorphisms into a set $\mathcal{K}_2$ of structures of the same type τ_2 closed under isomorphisms in the following way: F is isomorphism-preserving iff $\mathcal{X}F\mathcal{Y}$, $\mathcal{X}'F\mathcal{Y}'$ and $\varphi \in \mathrm{I}(\mathcal{X},\mathcal{X}')$ implies $\varphi \in \mathrm{I}(\mathcal{Y},\mathcal{Y}')$. The second part of the equivalence can be written as follows: If $\varphi \in \mathrm{I}(\mathcal{X},\varphi[\mathcal{X}])$ then $F(\varphi[\mathcal{X}]) = \varphi[F(\mathcal{X})]$.

Lemma 38 *If $F : \mathrm{C}_P \to \mathrm{C}_Q$ such that $P^Q = F$ then $Q = F \circ P$.*

Proof. Suppose that $P^Q = F$ and $F : \mathrm{C}_P \to \mathrm{C}_Q$. Then for all $A \in \mathcal{D}$,

$$P^Q(P(A)) = F(P(A)).$$

Since $P(A)\,P^Q PQ(A)$ and P^Q is a function it follows that $Q(A) = F(P(A))$ and, hence, $Q = F \circ P$. ■

Suppose $Q = F \circ P$. Since all functions are correspondences so is F, and we can of course state the equation as $Q = P|F$. But when a correspondence is a function we prefer $F \circ P$ instead of $P|F$.

Lemma 39 *If $F : \mathrm{C}_P \to \mathrm{C}_Q$ such that F is isomorphism-preserving and $Q = F \circ P$ then $P^Q = F$.*

Proof. Suppose $\mathcal{X} \in \mathrm{C}_P$. Then there is $A \in \mathcal{D}$ and such that $\varphi \in \mathrm{I}(P(A),\mathcal{X})$. Since $F(P(A)) = Q(A)$, $\varphi \in \mathrm{I}(P(A),\mathcal{X})$ and F is isomorphism-preserving

$$F(\mathcal{X}) = \varphi[Q(A)].$$

Since $P^Q(P(A)) = Q(A)$ and $\varphi \in \mathrm{I}(P(A),\mathcal{X})$ it follows that

$$P^Q(\varphi[P(A)]) = \varphi[Q(A)]$$

which means that $P^Q(\mathcal{X}) = F(\mathcal{X})$, and, hence, $P^Q(\mathcal{X}) = F(\mathcal{X})$. ■

Suppose that P is a relational with range of definition $\mathcal{D}$ and F is a function on C_P to a class $\mathcal{K}$ of structures of the same type τ closed under isomorphism and, furthermore, F is isomorphism-preserving. Then $Q = F \circ P$ *determines* a relational Q with range of definition $\mathcal{D}$ and $\mathrm{C}_Q = \mathcal{K}$ and $P^Q = F$.

4.3.5 Uncorrelation and the strength of dependence

Definition 40 *P is* uncorrelated *with Q, denoted Puncorr$Q$ and $P \nparallel Q$ if*

$$P^Q = \{\langle \mathcal{X}, \mathcal{Y} \rangle \in \mathrm{C}_P \times \mathrm{C}_Q \mid X = Y\}$$

and we say that the transition P^Q is sweeping. *If P is not uncorrelated with Q then P is correlated with Q, denoted $P \| Q$.*

Note that if $P \nparallel Q$ then $Q \nparallel P$, and if $P\|Q$ then $Q\|P$. Note further that if $P \Uparrow Q$ and $P(A) = \mathcal{A}$ then there is only one possible extension of Q over A and we can denote it $P^Q(\mathcal{A})$ since P^Q is a function. If $P \nparallel Q$ and $P(A) = \mathcal{A}$ then $\mathcal{A}P^Q\mathcal{X}$ holds for all $\mathcal{X}$ such that $\mathcal{X} \in \mathrm{C}_Q$ and $A = X$. Hence the following holds: If $P \Uparrow Q$ then P completely determines Q. If $P \nparallel Q$ then P does not determine Q at all. If $P\|Q$ then P determines Q to some degree. The strength of determination or dependence can be of different degrees. Note that determination is directed. P can completely determine Q while Q only partially determines P. The determination of Q by P is therefore not necessarily of the same strength as the determination of Q by P. Subordination and uncorrelation are in a sense endpoints on a dependence scale.

As was pointed out in subsection 4.3.2 $P \Uparrow Q$ means that Q is implicitly definable by P and P^Q is a representation of the implicit definition of Q from P. From this follows that complete determination is equivalent to implicitly definability. And, hence, partial determination is equivalent to partial implicitly definability. Note further: the "wider" P^Q is, the less dependent is Q on P. These remarks is intended to give an "informal characterization" of subordination, correlation and transition and must not be taken too literally.

4.3.6 From correspondences to set-valued functions

There are in some contexts simplifying to transform a correspondence to a function. The general method is presented in subsection 1.3.2 by introducing $\overrightarrow{\gamma}$. We use this method in two of the following three definitions. ($\wp(X)$ is the power set of X.)

Definition 41 $\overrightarrow{P^Q} : \mathrm{C}_P \to \wp(\mathrm{C}_Q)$ *such that*

$$\overrightarrow{P^Q}(\mathcal{X}) = P^Q[\mathcal{X}].$$

Definition 42 P_*^Q *is the correspondence from* D_P *to* C_Q *defined by*

$$AP_*^Q\mathcal{Y} \text{ iff } P(A)\,P_*^Q\mathcal{Y}.$$

Definition 43 $\overrightarrow{P_*^Q} : \mathrm{D}_P \to \wp(\mathrm{C}_Q)$ *such that*

$$\overrightarrow{P_*^Q}(A) = \overrightarrow{P^Q}(P(A)).$$

Suppose that $P \nparallel Q$ and $\mathcal{X} \in \mathrm{C}_P$ and $A \in \mathrm{D}_P$. Then

$$\overrightarrow{P^Q}(\mathcal{X}) = \left\{\mathcal{Y} \mid \mathcal{X}P^Q\mathcal{Y}\right\} = \{\mathcal{Y} \in \mathrm{C}_Q \mid X = Y\}$$

$$\overrightarrow{P_*^Q}(A) = \{\mathcal{Y} \in \mathrm{C}_Q \mid A = Y\}.$$

4.3.7 Relative product of transitions

The relative product of two correspondences $\left\langle C_P, C_Q, P^Q \right\rangle$ and $\left\langle C_Q, C_R, Q^R \right\rangle$ is the correspondence $\left\langle C_P, C_R, P^Q|Q^R \right\rangle$ where $P^Q|Q^R$ is defined by

$$P^Q|Q^R = \left\{ \langle \mathcal{X}, \mathcal{Y} \rangle \in C_P \times C_R \mid \exists \mathcal{Z} \in C_Q : \mathcal{X} P^Q \mathcal{Z} \ \& \ \mathcal{Z} P^Q \mathcal{Y} \right\}.$$

Theorem 44 *$P^Q|Q^R \supseteq P^R$ and if $Q \Uparrow P$ or $Q \Uparrow R$ then $P^Q|Q^R = P^R$.*

Proof. (I) We first prove $P^Q|Q^R \supseteq P^R$. Suppose that $\mathcal{X} P^R \mathcal{Y}$. Then there is $A \in D_P$ and $\varphi \in \mathrm{Bi}(X, A)$ such that

$$\varphi^{-1}[P(A)] = \mathcal{X}$$

$$\varphi^{-1}[R(A)] = \mathcal{Y}.$$

Let $Z = \varphi^{-1}[Q(A)]$. Since

$$\varphi^{-1}[P(A)]\, P^Q \varphi^{-1}[Q(A)]$$

$$\varphi^{-1}[Q(A)]\, Q^R \varphi^{-1}[R(A)]$$

it follows that $\mathcal{X} P^Q \mathcal{Z}$ and $\mathcal{Z} Q^R \mathcal{Y}$, which implies that $\mathcal{X} P^Q \mathcal{Z}$.

(II) Now suppose that $\mathcal{X} P^Q|Q^R \mathcal{Y}$. Then there is $\mathcal{Z}$ such that

$$\mathcal{X} P^Q Z \ \& \ \mathcal{Z} Q^R \mathcal{Y}.$$

Hence, there is $A, B \in \mathcal{D}$ and $\varphi \in \mathrm{Bi}(X, A)$, $\psi \in \mathrm{Bi}(Y, B)$ such that

$$\mathcal{X} = \varphi^{-1}[P(A)] \quad \mathcal{Z} = \varphi^{-1}[Q(A)]$$

$$\mathcal{Z} = \psi^{-1}[Q(B)] \quad \mathcal{Y} = \psi^{-1}[R(B)]$$

From this follows that

$$\varphi^{-1}[Q(A)] = \psi^{-1}[Q(B)]$$

and hence

$$\varphi \circ \psi^{-1}[Q(B)] = Q(A)$$

which implies that $\varphi \circ \psi^{-1} \in I_Q(B, A)$.

(i) Now suppose that $Q \precsim P$ from which follows that $I_Q \dot{\subseteq} I_P$. Hence $\varphi \circ \psi^{-1} \in I_p(B, A)$, which implies that

$$\varphi \circ \psi^{-1}[P(B)] = P(A).$$

Thus

$$\psi^{-1}[P(B)] = \varphi^{-1}[P(A)].$$

Note that

$$\psi^{-1}[P(B)]\, P^R \psi^{-1}[R(B)]$$

and hence

$$\varphi^{-1}[P(A)]\, P^R \psi^{-1}[R(B)]$$

and therefore

$$\mathcal{X} P^R \mathcal{Y}.$$

This proves that $Q \Uparrow P$ implies that $P^Q | Q^R = P^R$. See the illustration below.

(ii) Finally suppose that $Q \Uparrow R$ and hence $\mathrm{I}_Q \dot{\subseteq} \mathrm{I}_R$. Since $\varphi \circ \psi^{-1} \in \mathrm{I}_Q(B, A)$ it follows that $\varphi \circ \psi^{-1} \in \mathrm{I}_R(B, A)$ which implies that

$$\varphi \circ \psi^{-1} [R(B)] = R(A).$$

Thus

$$\psi^{-1} [R(B)] = \varphi^{-1} [R(A)].$$

Note that

$$\varphi^{-1} [P(A)] \, P^R \varphi^{-1} [R(A)]$$

and hence

$$\varphi^{-1} [P(A)] \, P^R \psi^{-1} [R(B)]$$

and therefore

$$\mathcal{X} P^R \mathcal{Y}.$$

This proves that $Q \Uparrow R$ implies that $P^Q | Q^R = P^R$.[14] ■

The following diagrams illustrate part of the theorem, were $\Uparrow$ illustrates superiority. (P^Q is a correspondence from C_P to C_Q and Q^R is a correspondence from C_Q to C_R. P^R as well as $P^Q | Q^R$ are correspondences from C_P to C_R.)

$$\begin{array}{ccc} & & R \\ & \nearrow & \\ P & & \uparrow_{\Uparrow} \\ & \searrow & \\ & & Q \end{array} \qquad \Longrightarrow \qquad P^Q | Q^R = P^R.$$

$$\begin{array}{ccc} P & & \\ & \searrow & \\ \downarrow_{\Uparrow} & & R \\ & \nearrow & \\ Q & & \end{array} \qquad \Longrightarrow \qquad P^Q | Q^R = P^R.$$

Relative product of transitions are correspondences and some results of applying transitions to structures (extensions) and sets of structures are shown below.

It holds generally that

$$(P^Q | Q^P) [\mathcal{X}] = Q^R \left[P^Q [\mathcal{X}] \right].$$

Since $P^Q | Q^R \supseteq P^R$ it follows that $P^R [\mathcal{X}] \subseteq Q^R \left[P^Q [\mathcal{X}] \right]$, i.e.

$$\overrightarrow{P^R} (\mathcal{X}) \subseteq Q^R \left[\overrightarrow{P^Q} (\mathcal{X}) \right].$$

[14] Cf. Odelstad (1992) Theorem 7.3.8–7.3.10.

Let $\mathcal{X} = P(A)$. Then

$$P^R[A] \subseteq Q^R\left[P^Q[A]\right]$$

and thus

$$\overrightarrow{P^R_*}(A) \subseteq Q^R\left[\overrightarrow{P^Q_*}(A)\right].$$

If $P^Q|Q^R = P^R$ then

$$P^R[\mathcal{X}] = Q^R\left[P^Q[\mathcal{X}]\right]$$

$$P^R[\mathcal{X}] = Q^R\left[\overrightarrow{P^Q}(\mathcal{X})\right]$$

and hence

$$P^R[A] = Q^R\left[P^Q_*[A]\right]$$

$$\overrightarrow{P^R_*}(A) = Q^R\left[\overrightarrow{P^Q_*}(A)\right].$$

4.3.8 Tightness

The notion ‘tightness’ defined below expresses a kind of dependence relation between relationals (cf. subsection 4.3.5).

Definition 45 *We say that P^R is* at least as tight *as Q^R, which is denoted $P^R \underline{\lessdot} Q^R$, if for all $A \in \mathrm{D}$*

$$\overrightarrow{P^R_*}(A) \subseteq \overrightarrow{Q^R_*}(A).$$

If $P^R \underline{\lessdot} Q^R$ and $Q^R \underline{\lessdot} P^R$ we denote it $P^R \doteq Q^R$ and say that P^R and Q^R are equally tight. *If $P^R \underline{\lessdot} Q^R$ but not $P^R \doteq Q^R$ we denote it $P^R \lessdot Q^R$ and say that P^R is* tighter *than Q^R.*

Note that $P^R \underline{\lessdot} Q^R$ means informally that R is more dependent on P than on Q (more determined by P than by Q).

Theorem 46 *Suppose that $P \Uparrow Q$. Then $P(A)\, P^R \mathcal{X}$ implies that $Q(A)\, Q^R \mathcal{X}$, i.e. for all $A \in \mathrm{D}$*

$$\overrightarrow{P^R_*}(A) \subseteq \overrightarrow{Q^R_*}(A)$$

in other words $P^R \underline{\lessdot} Q^R$.

The theorem is illustrated below.

$$\begin{array}{ccccc} Q & & & & \\ & \searrow & & & \\ \Uparrow & & R & \Longrightarrow & P^R \underline{\lessdot} Q^R \\ & \nearrow & & & \\ P & & & & \end{array}$$

Proof. Suppose that $P \Uparrow Q$. Then

$$Q^P|P^R = Q^R.$$

Suppose that $P(A)\, P^R \mathcal{X}$. This together with $Q(A)\, Q^P P(A)$ implies

$$Q(A)\left(Q^P | P^R\right)\mathcal{X}$$

and hence

$$Q(A)\, Q^R \mathcal{X}.$$

This shows that $\mathcal{X} \in \overrightarrow{P^R_*}(A)$ implies that $\mathcal{X} \in \overrightarrow{Q^R_*}(A)$, i.e. for all $A \in$ D

$$\overrightarrow{P^R_*}(A) \subseteq \overrightarrow{Q^R_*}(A)$$

and hence

$$P^R \underline{\lessdot} Q^R.$$

■

Theorem 47 *If $P^Q | Q^R = P^R$ then for all $A \in \mathcal{D}$,*

$$\overrightarrow{Q^R_*}(A) \subseteq \overrightarrow{P^R_*}(A)$$

i.e.

$$Q^R \underline{\lessdot} P^R.$$

Proof. Let $A \in \mathcal{D}$ arbitrary. Suppose that $\mathcal{A} \in \overrightarrow{Q^R_*}(A)$. Hence, $Q(A)\, Q^R \mathcal{A}$. Since $P(A)\, P^Q Q(A)$ it follows that

$$P(A)\left(P^Q | Q^R\right)\mathcal{A}.$$

Since $P^Q | Q^R = P^R$ it follows that $P(A)\, P^R \mathcal{A}$ and hence $\mathcal{A} \in \overrightarrow{P^R_*}(A)$. We have thus proved that $\mathcal{A} \in \overrightarrow{Q^R_*}(A)$ implies $\mathcal{A} \in \overrightarrow{P^R_*}(A)$, i.e.

$$\overrightarrow{Q^R_*}(A) \subseteq \overrightarrow{P^R_*}(A).$$

Since $A \in \mathcal{D}$ arbitrary it follows that $Q^R \underline{\lessdot} P^R$. ■

4.4 Transitions and joinings between relationals of different sorts

4.4.1 Introduction

This section is a first presentation of a work in progress on the aspect model of TJS. The TJS-framework is here used as a toolbox for an inquiry into the joining of systems of aspects of different sorts and, as the work proceeds, with focus on the function and structure of intermediate concepts. One important aspect of the inquiry is "conceptual openness", i.e. the feature of intermediaries being ground and/or consequence open. Such intermediaries can function as decision nodes in a step by step decision process where the "Spielraum" of the decision is partially determined by the grounds and consequences of the concepts and will be more and more restricted as the process proceeds. In multi-criteria analysis such step by step processes seem to be frequent and reasonable, and open intermediaries may play a role in decision support systems for multi-critera problems.

4.4.2 The implicative character of transitions

As was emphasized already in the subsection 1.2 it is essential for the TJS-perspective on conceptual structures that there is an implicative relation between the concepts. The characteristics of this implicative relation differ depending on the type and sort of the conceptual structure. In the *cis* model $a \precsim b$ represents in a sense the conditional statement '$\forall x : a(x) \rightarrow b(x)$' and $\precsim$ represent implication. We can of course see the conditions a and b as functions which assign truth values to $a(x)$ and $b(x)$ in case the conditional statement is represented as follows: 'If $a(x) = \top$ then $b(x) = \top$.' In the aspect model $P \Uparrow Q$ ($QsubP$) represents the statement that if the extension of P on a set A is given then the extension of Q on A is determined. However, from the extension of P on A does not follow the extension of Q on A, just that it is determined. Here the transitions enter the picture. Let P and Q be relationals with the same range $\mathcal{D}$ of definition. If $A \in \mathcal{D}$, $\mathcal{A} \in \mathrm{C}_P$ and $P(A) = \mathcal{A}$ then $Q(A) \in \overrightarrow{P^Q}(\mathcal{A})$, where $\overrightarrow{P^Q}(\mathcal{A}) = P^Q[\mathcal{A}]$. A somewhat more detailed formulation is the following. If $A \in \mathcal{D}$, $\mathcal{A} \in \mathrm{C}_P$, $P(A) = \mathcal{A}$ and $P^Q = \Gamma$ then $Q(A) \in \overrightarrow{\Gamma}(\mathcal{A})$, where $\langle \mathrm{C}_P, \mathrm{C}_Q, \Gamma \rangle$ is a correspondence closed under isomorphisms. If $P \Uparrow Q$ then Γ is an isomorphism-preserving function and $Q(A) = \Gamma(\mathcal{A})$ if $P(A) = \mathcal{A}$.

In the *cis* model the joinings between structures of different sorts are conditional statements (implications). In the aspect model the joinings can be represented as transitions and the same holds for the grounds and consequences of intermediaries. This line of thought will be developed below (see subsections 4.4.6–4.4.8).

4.4.3 Conditions as relationals

A condition can in many situations be regarded as a relational. Let a be a ν-ary condition on X. $a(X)$ is then the set of all elements in X^ν such that they satisfy a, i.e.

$$a(X) = \{\langle x_1, x_2, ..., x_\nu \rangle \in X^\nu | a(x_1, x_2, ..., x_\nu)\}.$$

Let a and b be conditions regarded as relationals with the same range of definition $\mathcal{D}$ and let $a \precsim b$ represents that a implies b. Suppose that $a \precsim b$. Then $a(X) \subseteq b(X)$ for all $X \in \mathcal{D}$ which we denote $a \dot{\subseteq} b$ (using pointwise definition, see subsection 4.3.2). The transition from a to b is denoted a^b. Then

$a \dot{\subseteq} b$ iff $\forall X \in \mathcal{D} : \forall \lambda, \mu \subseteq X^\nu : \langle X, \lambda \rangle a^b \langle X, \mu \rangle \Leftrightarrow \lambda \subseteq \mu$.

But suppose now that a and b are of different sorts and that there is a conditional norm $a \dot{\subseteq} b$ which holds according to a normative system. Then the transition from a to b, i.e. a^b, is a norm determined by the normative system. Transitions can therefore in some contexts represent conditional norms. And even if they do not represent conditional norms since the relationals involved are not conditions, the transitions can in some situations be of a normative character. This shows more generally that transitions can function as joinings between relational systems of different sorts.

4.4.4 Relational arrangements

As has been pointed out several times in this paper there are different sorts of conditions. The same holds for aspects. If $\mathfrak{R}$ is a set of relationals with the same range of definition and of the same sort, let us say σ, then we presuppose that for all $P, Q \in \mathfrak{R}$ the value of the transition P^Q "is determined", i.e. with regard to the transitions between relations of the same sort there is no indeterminacy.[15] Below we shall contrast this with contexts where indeterminacy exists. First a reminder. If $P||Q$, i.e. P and Q are correlated, then

$$P^Q \subset \{\langle \mathcal{X}, \mathcal{Y} \rangle \in \mathrm{C}_P \times \mathrm{C}_Q \mid X = Y\}$$

and P can be said to partially determine Q.

Suppose that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are sets of relationals of different sorts. Let $\mathfrak{R}_0 = \mathfrak{R}_1 \cup \mathfrak{R}_2$. If $P_1, Q_1 \in \mathfrak{R}_1$ and $P_2, Q_2 \in \mathfrak{R}_2$ then $P_1^{Q_1}$ and $P_2^{Q_2}$ are meaningful (determined) whereas this is not certainly the case for $P_1^{P_2}$ and $Q_1^{Q_2}$. If $P_1^{P_2}$ is determined such that $P_1||P_2$ then $P_1^{P_2}$ can represent a joining between $\mathfrak{R}_1$ and $\mathfrak{R}_2$ and be a part of the joined system $\mathfrak{R}_0$. The joining $P_1^{P_2}$ is especially strong if $P_1^{P_2}$ is a function. A special case of joining of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ is obtained if all the joinings between $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are functions. We will study this in some detail further below. The narrowness-relation in a protojoining-system represents a kind of implication relation between joinings. What this means in the case of aspects will also be set out below in subsection 4.4.6.

In order to simplify the presentation in the coming subsections the following notion is introduced. A set of relationals with the same range of definition is a *relational arrangement* if the transitions between all elements in $\mathfrak{R}$ are determined. A set of relations of the same sort and the same range of definition is therefore a relational arrangement.

4.4.5 Two derivation schemata

In the presentation of the *cis* model in subsection 1.2 two "derivation schemata" are mentioned. There are corresponding schemata for aspects. At first we take schema (II):

(II-A)
$Q_1 \Uparrow_1 P_1$
$\langle P_1, P_2 \rangle$ [joining]
$P_2 \Uparrow_2 Q_2$
————-
$\langle Q_1, Q_2 \rangle$ [joining]

In schema (I-A) the input is the extension of a relational. Suppose that $P_1(A) = \mathcal{A}$ and $\langle P_1, P_2 \rangle$ is a joining. What is the output? It may seem reasonable to propose that the output is $P_1^{P_2}[\mathcal{A}]$. But since P_1 and P_2 belong to different sorts it is not clear what $P_1^{P_2}$ means. $P_1^{P_2}$ is a kind of bridging or crossing transition, *cross-transition* for short, and the meaningfulness (and meaning) of such transitions is determined by the actual system of joinings. If joining $\langle P_1, P_2 \rangle$ is assigned a correspondence Γ (closed under isomorphisms) then $P_2(A) \in \Gamma(\mathcal{A})$ and if Γ is a function then $P_2(A) = \Gamma(\mathcal{A})$. The schema (I-A) can be stated:

(I-A)
$P_1(A) = \mathcal{A}$

[15] "Determined" will here not imply "known". Our knowledge can be uncertain.

$\langle P_1, P_2\rangle$ [joining to which Γ is assigned]

$P_2(A) \in \Gamma(\mathcal{A})$

We will return to cross-transition below.

4.4.6 Protojoining-systems of aspects

Suppose that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are sets of relationals with range of definition $\mathcal{D}$ but of different sorts. Then $\langle \mathfrak{R}_1, \Uparrow_1\rangle$ and $\langle \mathfrak{R}_2, \Uparrow_2\rangle$ are quasi-orderings. From a formal point of view $\Uparrow_1$ and $\Uparrow_2$ are different relations since they are defined for different sorts.[16] Let $\widehat{\trianglelefteq}_{1,2}$ be the narrowness-relation relative to $\langle \mathfrak{R}_1, \Uparrow_1\rangle$ and $\langle \mathfrak{R}_2, \Uparrow_2\rangle$. Hence, for $P_1, Q_1 \in \mathfrak{R}_1$ and $P_2, Q_2 \in \mathfrak{R}_2$

$$\langle P_1, P_2\rangle \widehat{\trianglelefteq}_{1,2} \langle Q_1, Q_2\rangle \Longleftrightarrow Q_1 \Uparrow_1 P_1 \& P_2 \Uparrow_2 Q_2.$$

Note that $\left\langle \mathfrak{R}_1 \times \mathfrak{R}_2, \widehat{\trianglelefteq}_{1,2}\right\rangle$ is a quasi-ordering.

Suppose that $\mathfrak{J}_{1,2} \subseteq \mathfrak{R}_1 \times \mathfrak{R}_2$ and let $\Theta_{1,2} = \langle \langle \mathfrak{R}_1, \Uparrow_1\rangle, \langle \mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}\rangle$ be a *pJs*. Let $\mathfrak{R}_0 = \mathfrak{R}_1 \cup \mathfrak{R}_2$, $\mathfrak{R}_1 \cap \mathfrak{R}_2 = \varnothing$ and

$$\upuparrows_0 = \Uparrow_1 \cup \mathfrak{J}_{1,2} \cup \Uparrow_2. \tag{3}$$

According to Theorem 9 $\langle \mathfrak{R}_0, \Uparrow_0\rangle$ is a quasi-ordering.

There are different kinds of aspect-type *pJs* like $\Theta_{1,2}$ and we will here throw some light upon this. We start off with a reminder. Suppose that $\mathfrak{R}$ is a set of relationals with range of definition $\mathcal{D}$. Then, according to Theorem 44, if $Q \Uparrow P$ or $Q \Uparrow R$ then $P^Q|Q^R = P^R$. Note further that if P^Q and Q^R are functions then $P^Q|Q^R = P^R$ and we can write it $P^R = Q^R \circ P^Q$.

If $P_1, Q_1 \in \mathfrak{R}_1$ then $P_1 \upuparrows_0 Q_1$ implies that $P_1 \Uparrow_1 Q_1$ and, hence, there is an isomorphism-preserving function (*ipf* for short) $F_1 : \mathrm{C}_{P_1} \to \mathrm{C}_{Q_1}$ such that $Q_1 = F_1 \circ P_1$. If $P_2, Q_2 \in \mathfrak{R}_2$ then $P_2 \upuparrows_0 Q_2$ implies that $P_2 \Uparrow_2 Q_2$ and, hence, there is an *ipf* $F_2 : \mathrm{C}_{P_2} \to \mathrm{C}_{Q_2}$ such that $Q_2 = F_2 \circ P_2$. Consider now the following condition:

(*) If $P_1 \in \mathfrak{R}_1$, $P_2 \in \mathfrak{R}_2$ and $P_1 \upuparrows_0 P_2$ then there is an *ipf* $F_0 : \mathrm{C}_{P_1} \to \mathrm{C}_{P_2}$ such that $P_2 = F_0 \circ P_1$.

This condition does not follow from the assumption that $\Theta_{1,2}$ is a *pJs*. But it is of course possible that it holds for some *pJs's*, but in these cases $\Theta_{1,2}$ results in a much stronger joining of $\langle \mathfrak{R}_1, \Uparrow_1\rangle$ to $\langle \mathfrak{R}_2, \Uparrow_2\rangle$ then that $\Theta_{1,2}$ is just a *pJs*. Note that the strength of this kind of joining system is of another kind than being a *preJs* or a *Js*. But for (*) to express an interesting kind of joining of $\langle \mathfrak{R}_1, \Uparrow_1\rangle$ to $\langle \mathfrak{R}_2, \Uparrow_2\rangle$ something more is needed. Suppose that $P_1, Q_1, R_1 \in \mathfrak{R}_1$, $P_1 \Uparrow_1 Q_1$ and $Q_1 \Uparrow_1 R_1$. Then $P_1^{Q_1}$ and $Q_1^{R_1}$ are *ipf's* and $P_1^{R_1} = P_1^{Q_1}|Q_1^{R_1} = Q_1^{R_1} \circ P_1^{Q_1}$. And the same also holds for $\mathfrak{R}_2$. Suppose now that $P_1 \in \mathfrak{R}_1$, $P_2 \in \mathfrak{R}_2$ and there is an *ipf* $F_0 : \mathrm{C}_{P_1} \to \mathrm{C}_{P_2}$ such that $P_2 = F_0 \circ P_1$. Suppose that $Q_1 \in \mathfrak{R}_1$, $Q_2 \in \mathfrak{R}_2$ and that there is an *ipf* $G_0 : \mathrm{C}_{Q_1} \to \mathrm{C}_{Q_2}$ such that $Q_2 = G_0 \circ Q_1$. Let us now represent $P_1^{P_2} = F_0$ and $Q_1^{Q_2} = G_0$. Then

$$F_0 = P_1^{P_2} = P_1^{Q_1}|Q_1^{Q_2}|Q_2^{P_2} = Q_2^{P_2} \circ Q_1^{Q_2} \circ P_1^{Q_1} = Q_2^{P_2} \circ G_0 \circ P_1^{Q_1}.$$

---

[16]Note that $\Uparrow$ is in fact a relational with $\Uparrow_1$ and $\Uparrow_2$ as its extensions on different domains.

This shows the dependence between the *ips*'s associated to different elements $\langle P_1, P_2\rangle$ and $\langle Q_1, Q_2\rangle$ in $\upuparrows_0$.

Conditions as the above lie behind the following definition, where $\upuparrows_0$ is defined as above (see equation (3)):

Definition 48 $\Theta_{1,2} = \langle\langle\mathfrak{R}_1, \Uparrow_1\rangle, \langle\mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}, \Phi\rangle$ *is a* sub relational pJs (sr-pJs) *if* $\langle\langle\mathfrak{R}_1, \Uparrow_1\rangle, \langle\mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}\rangle$ *is a* pJs *and* Φ *is a function that assigns an* ips *to each* $\langle P, Q\rangle$ *in* $\upuparrows_0$ *such that*

$$\text{(i)}\ \ \textit{For } i = 1, 2 : P_i, Q_i \in \mathfrak{R}_i \Rightarrow \Phi(P_i, Q_i) = P_i^{Q_i}.$$

$$\text{(ii)}\ \ \langle P, Q\rangle, \langle Q, R\rangle \in \upuparrows_0 \Rightarrow \Phi(Q, R) \circ \Phi(P, Q) = \Phi(P, R).$$

Note that the joinings in a *sr-pJs*, i.e. the elements in $\mathfrak{J}_{1,2}$, are functions and we call them *functional joinings*. It is intuitively appealing to denote $\Phi(P_1, P_2)$ as $P_1^{P_2}$ and in that way view $\Phi(P_1, P_2)$ as a transition but of a special kind, viz. a cross-transition. In many contexts cross-transitions represent norms. But it must be remembered that a cross-transition $P_1^{P_2}$ of this kind is only meaningful if $\langle P_1, P_2\rangle \in \mathfrak{J}_{1,2}$ and the meaning of $P_1^{P_2}$ is relative to Φ. Note that $\widehat{\trianglelefteq}_{1,2}$ is a quasi-ordering on the set of cross-transitions in $\Theta_{1,2}$. Suppose that $\langle P_1, P_2\rangle, \langle Q_1, Q_2\rangle \in \mathfrak{J}_{1,2}$ and $\langle P_1, P_2\rangle \widehat{\trianglelefteq}_{1,2} \langle Q_1, Q_2\rangle$. Then $Q_1 \Uparrow_1 P_1$, $P_2 \Uparrow_2 Q_2$ and

$$Q_1^{P_1} | \Phi(P_1, P_2) = \Phi(Q_1, P_2) \ \text{ i.e. } P_1^{P_2} \circ Q_1^{P_1} = Q_1^{P_2}.$$

Further,

$$P_2^{Q_2} | \Phi(Q_1, P_2) = \Phi(Q_1, Q_2) \ \text{ i.e. } P_2^{Q_2} \circ Q_1^{P_2} = Q_1^{Q_2}.$$

and, hence,

$$P_2^{Q_2} \circ P_1^{P_2} \circ Q_1^{P_1} = Q_1^{Q_2}. \tag{4}$$

Note that (4) expresses a relationship between the functional joinings in a *sr-pJs*.

There are at least two different approaches to the study of *sr-pJs's.* In one approach the starting point is a given *sr-pJs* and the questions are how it is constructed, what is its content and its implications and so on. For the other approach the main question is to construct a *sr-pJs* from quasi-orderings of aspects, for example by the way of specifying a set of joinings and the Φ-values for those. For both these approaches the set of minimal joinings will play an important role, and we will make some observations on this problem.

The notion of connectivity can of course be applied to *sr-pJs's.* Let $\Theta_{1,2}$ be a *sr-pJs* specified as in Definition 48 and satisfying connectivity with respect to $\widehat{\trianglelefteq}_{1,2}$. Suppose that $Q_1 \in \mathfrak{R}_1$, $Q_2 \in \mathfrak{R}_2$ and $\langle Q_1, Q_2\rangle \in \upuparrows_0$. Then there is $\langle P_1, P_2\rangle \in \min \mathfrak{J}_{1,2}$ such that

$$\langle P_1, P_2\rangle \widehat{\trianglelefteq}_{1,2} \langle Q_1, Q_2\rangle$$

and by using (4) we get

$$P_2^{Q_2} \circ P_1^{P_2} \circ Q_1^{P_1} = Q_1^{Q_2}.$$

Suppose now that $\langle R_1, R_2\rangle \in \min \mathfrak{J}_{1,2}$ such that

$$\langle R_1, R_2\rangle \widehat{\trianglelefteq}_{1,2} \langle Q_1, Q_2\rangle.$$

Then

$$Q_1^{Q_2} = R_2^{Q_2} \circ R_1^{R_2} \circ Q_1^{R_1}$$

and, hence,

$$P_2^{Q_2} \circ P_1^{P_2} \circ Q_1^{P_1} = R_2^{Q_2} \circ R_1^{R_2} \circ Q_1^{R_1}.$$

This shows that it is not possible to choose the Φ-value for every element in $\min \mathfrak{J}_{1,2}$ separately, independently of the Φ-values for the other elements in $\min \mathfrak{J}_{1,2}$. This depends on the structure on $\langle \min \mathfrak{J}_{1,2}, \upuparrows_0 \rangle$ (see Lindahl & Odelstad, 2013, p. 588 Corollary 3.36, cf. subsection 1.3.2 above) but this problem area will be left for future research.

4.4.7 Non-functional cross-transitions

A transition that is a function will below be called a *functional transition* and a transition which is not a function will be called *non-functional*. In the subsections 4.4.6 above it was assumed that a joining is a functional cross-transition. However, a non-functional cross-transition that does not imply uncorrelation can be considered as a partial or open joining (an open norm if normativity is involved) determining a *Spielraum* of consequences.[17] To what extent the non-functional cross-transitions are determined by the functional ones in a *sr-pJs* will be discussed in this subsection.

Let $\Theta_{1,2}$ be a *sr-pJs* specified as in Definition 48 above. Then $\mathfrak{J}_{1,2}$ determines an extension of $\langle \mathfrak{R}_1, \Uparrow_1 \rangle$ and $\langle \mathfrak{R}_2, \Uparrow_2 \rangle$ to the quasi-ordering $\langle \mathfrak{R}_0, \upuparrows_0 \rangle$ in an unique way. Note that if $\langle P, Q \rangle \in \upuparrows_0$ then the transition P^Q is, as explained above, meaningfully characterized and can be regarded as a primary transition. We will here discuss the possibility to extend the set of meaningful transitions. Suppose that $Q_1 \in \mathfrak{R}_1$ and $Q_2 \in \mathfrak{R}_2$ such that there is no $R_2 \in \mathfrak{R}_2$ such that $\langle Q_1, R_2 \rangle \in \mathfrak{J}_{1,2}$ and no $R_1 \in \mathfrak{R}_1$ such that $\langle R_1, Q_2 \rangle \in \mathfrak{J}_{1,2}$. Neither Q_1 nor Q_2 is thus a component in a joining. We have not, so far, given any meaning to $Q_1^{Q_2}$. Suppose now that $\langle P_1, P_2 \rangle \in \mathfrak{J}_{1,2}$ which implies that there is an *ipf* F such that $P_1^{P_2} = F$. $Q_1^{P_1}$ is a transition in $\mathfrak{R}_1$ and $P_2^{Q_2}$ a transition in $\mathfrak{R}_2$ and, hence, there is a correspondence Γ_1 closed under isomorphisms such that $Q_1^{P_1} = \Gamma_1$ and a correspondence Γ_2 closed under isomorphisms such that $P_2^{Q_2} = \Gamma_2$. We can construct the correspondence $\Gamma_1|F|\Gamma_2$ which is a correspondence closed under isomorphism with C_{Q_1} as domain and C_{Q_2} as codomain. Since $\Gamma_1|F|\Gamma_2 = Q_1^{P_1}|P_1^{P_2}|P_2^{Q_2}$ there is a relation between $Q_1^{Q_2}$ and $\Gamma_1|F|\Gamma_2$. But it is not certain that we can define $Q_1^{Q_2} = \Gamma_1|F|\Gamma_2$. To see this, suppose that $\langle R_1, R_2 \rangle \in \mathfrak{J}_{1,2}$ and that $Q_1^{R_1} = \Gamma_1^*$, $R_1^{R_2} = F^*$ and $R_2^{Q_2} = \Gamma_2^*$. Then $\Gamma_1^*|F^*|\Gamma_2^* = Q_1^{Q_2}$. It is not certain that $\Gamma_1|F|\Gamma_2 = \Gamma_1^*|F^*|\Gamma_2^*$, not even if $\Theta_{1,2}$ satisfies connectivity and

$$\langle P_1, P_2 \rangle, \langle R_1, R_2 \rangle \in \min_{\trianglelefteq_{1,2}} \mathfrak{J}_{1,2}.$$

Instead it seems reasonable that

$$Q_1^{Q_2} \subseteq \Gamma_1|F|\Gamma_2 \ \& \ Q_1^{Q_2} \subseteq \Gamma_1^*|F^*|\Gamma_2^*$$

[17] Note that a non-functional transition P^Q can be "transformed" to $\overrightarrow{P^Q}$ which is a function assigning sets of values to the possible extension of Q over A given the extension of P over A.

and we see these statements as approximations of $Q_1^{Q_2}$. With a slight extension of the notion of tightness (see Definition 45) we say that $\Gamma_1|F|\Gamma_2$ is at least as tight as $\Gamma_1^*|F^*|\Gamma_2^*$, which is denoted

$$\Gamma_1|F|\Gamma_2 \underline{\lessdot} \Gamma_1^*|F^*|\Gamma_2^*,$$

if the following holds: If $\mathcal{X}_1 \in \mathrm{C}_{Q_1}$, $\mathcal{X}_2 \in \mathrm{C}_{Q_2}$ and $X_1 = X_2$ then

$$\mathcal{X}_1\,(\Gamma_1|F|\Gamma_2)\,\mathcal{X}_2 \Rightarrow \mathcal{X}_1\,(\Gamma_1^*|F^*|\Gamma_2^*)\,\mathcal{X}_2.$$

Different approximations of $Q_1^{Q_2}$ may differ in tightness and the optimization of one transition can affect what is a possible optimization of other transitions. The conditions under which there is a tightest approximation of $Q_1^{Q_2}$ and under which this is compatible with the tightest approximation of all other cross-transitions will here be left open for future research. Instead we will shift focus somewhat and consider the case where all cross-transitions are meaningful. We take the following definition as a starting point.

Definition 49 $\Theta_{1,2} = \langle\langle\mathfrak{R}_1, \Uparrow_1\rangle, \langle\mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}, \Phi, \langle\mathfrak{R}_0, \Uparrow_0\rangle\rangle$ *is an* extensively relational pJs (er-pJs) *if* $\langle\langle\mathfrak{R}_1, \Uparrow_1\rangle, \langle\mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}, \Phi\rangle$ *is a* sr-pJs *and* $\mathfrak{R}_0$ *is a relational arrangement such that*

1. $\mathfrak{R}_0 = \mathfrak{R}_1 \cup \mathfrak{R}_2$
2. $\Uparrow_0 = \Uparrow_1 \cup \mathfrak{J}_{1,2} \cup \Uparrow_2$
3. *If* $P_i, Q_i \in \mathfrak{R}_i$ *for* $i = 1, 2$ *then the transition* $P_i^{Q_i}$ *in* $\mathfrak{R}_0$ *agree with the transition* $P_i^{Q_i}$ *in* $\mathfrak{R}_i$.
4. *If* $\langle P_1, P_2\rangle \in \mathfrak{J}_{1,2}$ *then* $P_1^{P_2} = \Phi(P_1, P_2)$.

Let $\Theta_{1,2}$ be a *er-pJs* specified as in Definition 49 above. With a cross-transition in $\mathfrak{R}_0 = \mathfrak{R}_1 \cup \mathfrak{R}_2$ is meant a transition P^Q such that either $P \in \mathfrak{R}_1$ and $Q \in \mathfrak{R}_2$ or $P \in \mathfrak{R}_2$ and $Q \in \mathfrak{R}_1$. $\mathfrak{J}_{1,2}$, the set of joinings in $\Theta_{1,2}$, consists of functional cross-transitions. Suppose that $P_1 \in \mathfrak{R}_1$ and $P_2 \in \mathfrak{R}_2$ such that $P_1^{P_2}$ is a non-functional cross-transition. Then it is possible that the correspondence Γ such that $P_1^{P_2} = \Gamma$ may not be inferred from $\mathfrak{J}_{1,2}$ and $P_1^{P_2} = \Gamma$ may be informative. $P_1^{P_2}$ is then a kind of partial (vague, open) joining from $\langle\mathfrak{R}_1, \Uparrow_1\rangle$ to $\langle\mathfrak{R}_2, \Uparrow_2\rangle$.

In $\mathfrak{R}_0$ all transitions, even cross-transitions, are meaningful. Hence, if $P_1 \in \mathfrak{R}_1$ and $P_2 \in \mathfrak{R}_2$ then $P_1^{P_2}$ is meaningful. However, that does not imply that $P_1^{P_2}$ is a proper joining, not even a proper partial joining. If $P_1^{P_2}$ is sweeping, i.e. $P_1 uncorr P_2$, then information on the extension of P_1 on A contains no information on the extension of P_2 on A. To be a proper joining $P_1^{P_2}$ must be non-sweeping, i.e. P_1 and P_2 must be correlated.

There is some vagueness in Definition 49 since the notion of a relational arrangement has not been given a formal characterization (for example by an axiomatization) that is satisfactory for the application of the notion in the actual context. This shortcoming will hopefully be possible to remedy with a deeper study of the structure of transitions.[18]

At this point some rules regulating the behavior of transitions ought to be remembered, see Theorem 44:

(1) Generally: $P^Q|Q^R \supseteq P^R$

(2) If $Q \Uparrow P$ then $P^Q|Q^R = P^R$

(3) If $Q \Uparrow R$ then $P^Q|Q^R = P^R$.

Of another kind but still related is the following (see Definition 45 and Theorem 46):

(4) If $P \Uparrow Q$ then $P^R \underline{\lessdot} Q^R$.

4.4.8 Non-functional joinings

Let $\Theta_{1,2} = \langle\langle\mathfrak{R}_1, \Uparrow_1\rangle, \langle\mathfrak{R}_2, \Uparrow_2\rangle, \mathfrak{J}_{1,2}\rangle$ be a *pJs*. We have so far studied the case where $\mathfrak{J}_{1,2}$ consists of functions. This is the case if $\Theta_{1,2}$ is an *er*- or *sr-pJs*, but it is also possible that $\mathfrak{J}_{1,2}$ consists of non-functional correspondences or a mixture of functional and non-functional correspondences. The narrowness-relation for such a *pJs* is the relation $\widehat{\trianglelefteq}_{1,2}$ defined above in terms of $\Uparrow_1$ and $\Uparrow_2$. If one accepts to determine approximative values for cross-transitions that are not joinings from the joinings in $\mathfrak{J}_{1,2}$ as was discussed in subsection 4.4.7, these can be regarded as a kind of secondary joinings while the elements in $\mathfrak{J}_{1,2}$ are primary joinings.

Suppose that $\Theta_{1,2}$ satisfies connectivity and that $\langle Q_1, Q_2\rangle \in \mathfrak{J}_{1,2}$. Then there is $\langle P_1, P_2\rangle \in \min_{\widehat{\trianglelefteq}_{1,2}} \mathfrak{J}_{1,2}$ such that

$$\langle P_1, P_2\rangle \,\widehat{\trianglelefteq}_{1,2}\, \langle Q_1, Q_2\rangle \quad \text{i.e.} \quad Q_1 \Uparrow_1 P_1 \& P_2 \Uparrow_2 Q_2.$$

It is often convenient to denote the elements in $\mathfrak{J}_{1,2}$ not as ordered pairs $\langle P_1, P_2\rangle$ but as transitions $P_1^{P_2}$. Suppose that Γ is a correspondence closed under isomorphism (and satisfying some other minor conditions, see subsection 4.3.3) such that $P_2 = P_1|\Gamma$. Let us for the sake of simplicity accept the "transition-formalism" and, hence, $P_1^{P_2} = \Gamma$ and

$$Q_1^{Q_2} \subseteq Q_1^{P_1}|\Gamma|P_2^{Q_2}.$$

Hence, the set of minimal elements does not necessarily determine all the elements in $\mathfrak{J}_{1,2}$.

If $\mathfrak{J}_{1,2}$ consists of correspondences that are not certainly functions then the connection between $\langle\mathfrak{R}_1, \Uparrow_1\rangle$ and $\langle\mathfrak{R}_2, \Uparrow_2\rangle$ seems to be in a sense unspecified or vague. But this vagueness can be the result of lack of knowledge about the joined system and may be settled at a later stage of the investigation. A *pJs* with correspondences as joinings can be a useful tool in the initial stage of a research about a *pJs* of relationals but also a tool for constructing a more developed joined system.

[18] In a *pJs* $\langle\mathcal{A}_1, A_2, J\rangle$ there is often more "structure" on the top $\mathcal{A}_1$ and the bottom A_2 than just the quasi-orderings. In the *cis* model the top and bottom are Boolean quasi-orderings. In a system of relations there is also "more structure" than just the quasi-orderings, for example the concatenation operation of relationals, that from P and Q form PQ (see Definition 31). In some applications we can have a set of "generators" G such that each relational in G is uncorrelated to the concatenation of all the other elements in G. For all finite subsets H of G we construct the concatenation of the element in H. The resulting set of relationals (G included) can in a sense be regarded as a set of relationals generated by G. However, structures on sets of relationals can be specified in other ways, too.

4.4.9 Relationals as intermediaries

In the subsections above some aspects of the connection between functional and non-functional cross transitions from relationals of one sort to relationals of another sort has been discussed. Some light has been shed upon the following questions. To what extent are the non-functional cross-transitions determined by the functional ones? And to what degree are the functional cross-transitions determined by the non-functional ones? These questions are relevant in connection with intermediaries. Can an intermediary have a non-functional cross-transition as the defining ground and/or consequence? Is that what "openness" means for aspects as intermediaries? Does this introduce indeterminacy in the conceptual system?

Suppose that

$$\Theta_{1,2} = \left\langle \langle \mathfrak{R}_1, \Uparrow_1 \rangle, \langle \mathfrak{R}_2, \Uparrow_2 \rangle, \mathfrak{J}_{1,2}, \Phi_{1,2}, \left\langle \mathfrak{R}_{1,2}, \Uparrow_{1,2} \right\rangle \right\rangle$$

$$\Theta_{2,3} = \left\langle \langle \mathfrak{R}_2, \Uparrow_2 \rangle, \langle \mathfrak{R}_3, \Uparrow_3 \rangle, \mathfrak{J}_{2,3}, \Phi_{2,3}, \left\langle \mathfrak{R}_{2,3}, \Uparrow_{2,3} \right\rangle \right\rangle$$

$$\Theta_{1,3} = \left\langle \langle \mathfrak{R}_1, \Uparrow_3 \rangle, \langle \mathfrak{R}_3, \Uparrow_3 \rangle, \mathfrak{J}_{1,3}, \Phi_{1,3}, \left\langle \mathfrak{R}_{1,3}, \Uparrow_{1,3} \right\rangle \right\rangle$$

are extensively relational *pJs's (er-pJs's)*. We represent elements in $\mathfrak{J}_{i,j}$ as cross-transitions. Let us suppose that $\Theta_{1,2}$, $\Theta_{2,3}$ and $\Theta_{1,3}$ are interconnected in such a way that if $P_1 \in \mathfrak{R}_1$, $P_2 \in \mathfrak{R}_2$ and $P_3 \in \mathfrak{R}_3$ then $P_1^{P_2}|P_2^{P_3} \supseteq P_1^{P_3}$.

Let us now introduce intermediaries in the aspect model as follows: $P_2 \in \mathfrak{R}_2$ is an *intermediary* in $\langle \Theta_{1,2}, \Theta_{2,3}, \Theta_{1,3} \rangle$ if there are correspondences $\Gamma_{1,2}$ and $\Gamma_{2,3}$ closed under isomorphisms such that the meaning of P_2 is given by $P_1^{P_2} = \Gamma_{1,2}$ and $P_2^{P_3} = \Gamma_{2,3}$ taken together as a whole, and

$$P_1^{P_3} = \Gamma_{1,2}|\Gamma_{2,3}.$$

In other words, $P_1^{P_2} = \Gamma_{1,2}$ (i.e. $P_2 = P_1|\Gamma_{1,2}$) is the defining ground of P_2 and $P_2^{P_3} = \Gamma_{2,3}$ (i.e. $P_3 = P_2|\Gamma_{2,3}$) is the defining consequence of P_2. If $P_1(A) = \mathcal{A}$ then

$$P_2(A) \in \overrightarrow{P_1^{P_2}}(\mathcal{A}) \ \ \& \ \ P_3(A) \in \overrightarrow{P_2^{P_3}}\left[\overrightarrow{P_1^{P_2}}(\mathcal{A})\right].$$

Suppose that $P_2 \in \mathfrak{R}_2$ is an intermediary in $\langle \Theta_{1,2}, \Theta_{2,3}, \Theta_{1,3} \rangle$ with $P_1^{P_2}$ as the defining ground and $P_2^{P_3}$ as the defining consequence, which we express by saying that the meaning of P_2 is given by $P_2 =_{df} P_1^{P_2}|P_2^{P_3}$. Then there are four different combinations of functional/non-functional grounds and consequences:

1. $P_1^{P_2}$ and $P_2^{P_3}$ are both functions; P_2 is neither ground nor consequence open
2. $P_1^{P_2}$ is a function but $P_2^{P_3}$ is not a function; P_2 is consequence open
3. $P_1^{P_2}$ is not a function but $P_2^{P_3}$ is a function, P_2 is ground open
4. neither $P_1^{P_2}$ nor $P_2^{P_3}$ are functions, P_2 is ground and consequence open.

The four items above constitutes the initial fragment of a typology of intermediaries.

If

$$P_2 =_{df} P_1^{P_2} | P_2^{P_3} \ \& \ P_1^{P_2} = \Gamma_{1,2} \ \& \ P_2^{P_3} = \Gamma_{2,3}$$

then $P_1^{P_3} = \Gamma_{1,2} | \Gamma_{2,3}$ is a condition on $\Theta_{1,3}$. The use of intermediaries is thus one possible method for characterizing, completely or partially, the joining of two sets of relationals of different sorts

There are two different perspectives on the role of intermediaries in the joining of systems: The aim of the investigation is (1) the conceptual analysis of the intermediary or (2) the construction of a system that joins two sets of relationals of different sorts and the intermediary is a vehicle for establishing joinings. If the aim is (1) then the joined system is the starting point, whereas if the aim is (2) the intermediary is the starting point.

If $P_2 =_{df} P_1^{P_2} | P_2^{P_3}$ and, furthermore, $P_1^{P_2} = F_{1,2}$ and $P_2^{P_3} = F_{2,3}$ where $F_{1,2}$ and $F_{2,3}$ are *ips's*, then there is no openness with regard to P_2, i.e. if $P_1(A) = \mathcal{A}$ then $P_3(A) = F_{2,3}(F_{1,2}(\mathcal{A}))$. But some kind of uncertainty may still play a role. It may be the case that we cannot determine $P_1(A)$ directly but know that $Q_1(A) = \mathcal{A}$, and therefore, $P_1(A) \in \overrightarrow{Q_1^{P_1}}(\mathcal{A})$ and

$$P_2(A) \in F_{1,2}\left[\overrightarrow{Q_1^{P_1}}(\mathcal{A})\right] \ \& \ P_3(A) \in F_{2,3}\left[F_{1,2}\left[\overrightarrow{Q_1^{P_1}}(\mathcal{A})\right]\right].$$

4.5 Conclusion

The aim of this paper is to further develop TJS in some respects and widen the range of application of the theory. We have seen how the idea of norms as ordered pairs is flexible enough to handle nested implications and hypothetical consequences. Tools from Formal Concept Analysis may be useful in TJS since formal concepts and minimal joinings are shown to be closely related. An aspect model of TJS has been outlined. All three investigations can be pursued further, but especially the application of TJS on aspects may hopefully develop TJS and result in useful practical methods and tools. Cross-transitions can in normative contexts be regarded as norms, partial (vague, open) if the cross transitions are not functional. A problem area not touched upon in this paper but presumably of great relevance is the numerical representation of aspects, for example by means of measurement, since measures can represent (and in a sense become) intermediaries. We end this paper with the conjecture that the aspect model of TJS may be a useful tool for the analysis of the notion of supervenience.

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