Toward a Systematization of Logics for Monadic and Dyadic Agency & Ability, Revisited [1] [2]

Paul McNamara

Abstract

I specify a very large class of logics with monadic and dyadic modal operators, primarily (but not exclusively) intended to represent monadic and dyadic agency in the tradition of Kanger, Pörn, Elgesem, etc. I explore logics both for pure monadic agency, pure dyadic agency, and mixed monadic-dyadic agency. Employing neighborhood semantic frames, but with an extra parameter governed by a modest algebraic structure, I prove determination theorems for all the consistent logics of those specified. I briefly present some motivation and rationales for some of the principles, but the main focus is on the framework and key meta-theorems.

1 Introduction

I prove a fundamental theorem for canonical models for logics developed for representations of agency broadly in the tradition of Kanger, Pörn, Elgesem, and others.[3] Monadic and dyadic agency logics are explored in both pure forms as well as

[1] An earlier version of this paper appeared in McNamara, Paul 2018. See note 8 for the main differences.

[2] This work owes a special debt to Lou Goble. Lou generously provided substantial comments that proved invaluable, as well as showing me that a somewhat complex algebraic structure for a parameter for propositions that I had proposed was overkill for proving completeness, although it did turn out that a more modest version of the algebraic structure was needed for soundness and thus for determination theorems. Some further particulars are mentioned below. Xavier Parent's comments led to my revisiting Elgesem's dissertation (Elgesem, 1993) and realizing that I had overlooked the second less obvious place in his dissertation where he takes the subject of *dyadic* agency back up (Chapter 3, section II.2). I also benefited indirectly from work on counts-as conditionals by Jones and Parent (Jones, Andrew and Xavier Parent 2007), and from discussion of principle K's plausibility for agency with Andrew Jones, Mark Brown, Risto Hilpinen, and Marek Sergot. I benefited from illuminating discussion of system EK with Guram Bezhanishvili. I am also indebted to three anonymous referees for DEON 18 for comments on an earlier version of this paper, each of whom provided helpful comments, and special thanks go to Mark Brown for extremely generously detailed comments on earlier drafts. Lastly, I am indebted to Sarah Blagdon and Hein Duijf for helpful suggestions.

[3] Kanger, Stig and Helle Kanger 1966/2001, Kanger, Stig 1972; Pörn, Ingmar 1970; Pörn, Ingmar 1977; Pörn, Ingmar 1989; Lindahl, Lars 1977; Elgesem, Dag 1993; Elgesem, Dag 1997; Jones, Andrew and Marek Sergot 1993, Jones, Andrew and Marek Sergot 1996; Santos, Filipe and Jose Carmo 1996; Santos, Filipe, Andrew Jones et al. 1997; Jones, Andrew and Xavier Parent 2007; Sergot, Marek 1999; Sergot, Marek 2013. In Governatori, Guido and Antonino Rotolo 2005, it is shown that an oft-cited basic monadic system for agency and ability is incomplete on Elgesem's semantics, and the authors then go on to prove completeness for that system extended with one axiom, both using Elgesem's semantics and that of standard neighborhood semantics. The work here, as a consequence of the core theorems within, entails completeness for a large class of such monadic agency logics, and goes on to do the same for logics for dyadic agency and dyadic-monadic agency, dyadic agency being a much-underexplored area generally in this tradition (and others employing agential operators). There has been no systematic metatheory for monadic much less dyadic

mixed forms (monadic agency logics, dyadic agency logics, and monadic-dyadic agency logics). Special interest is given to *dyadic* agency. Elgesem is the only author in this tradition who explored dyadic agency. In my judgment, this constitutes a gap in this tradition, especially in the area of systematic exploration of these logics.

In places where I list formulae that to my knowledge have not been considered in this tradition, or considered and rejected perhaps prematurely, I will remark on those in footnotes. Otherwise, the focus will be on proving a fundamental theorem and then proving a variety of correspondence theorems for formulae and frame constraints, which will entail a very large number of strong completeness results. I then prove basic theorems entailing soundness results for these logics (save any inconsistent ones), and so collectively, a very large number of determination theorems follow, thus providing the first systematic account of such agency logics. Because I will consider some formulae and systems that are not usually considered for agency or ability, or for modal logic completeness using neighborhood frames (e.g. K, and system EK respectively), and for which completeness proofs appear to stall unless facilitated by some technical ploy like that used herein (or by using secondary canonical models devised on the fly), the model structures employed here will be slightly non-standard neighborhood frames.[4] There will be one extra parameter, P for propositions, and some of the frame constraints will then be relativized to that parameter in fairly innocuous ways to facilitate basic theorems entailing copious strong completeness proofs. I will explain the role of P as the issue comes up, and indicate where I believe it is manifested in the proofs below, since the frames are slightly unusual, as are a few of the constraints. Appendix 1 provides a quick preview. I also impose a modest algebraic structure on the

agency logics using neighborhood semantics or minor variants thereof. Here I approach the subject systematically, closing gaps in the monadic work, and dyadic work and I do this via a fundamental theorem for canonical models for a large class of such logics.

[4] Appendix 1 contains a short email with such a stalled proof and then a "fixed" proof. This email or a similar version of it was shared with a number of logicians, including Brian Chellas, Eric Pacuit, Steven Kuhn, Ed Mares, Lou Goble, Xavier Parent, and various logicians at Bayreuth and Ghent, and the basic problem was outlined at *Trends in Logic* (McNamara, Paul 2017). There was no indication that the stalling of the apparent straightforward attempt at a correspondence proof for K, nor a solution (that would yield full completeness for say EK with a standard semantic neighborhood clause for K), was known. Pacuit, Eric 2017 does not mention the problem with K nor attempt completeness for EK, nor is it discussed in Arlo-Costa, Horacio and Eric Pacuit 2007, although in both places completeness for the *normal* modal logic K (i.e. EMCN) is discussed. Chellas leaves a proof of EK's completeness as an exercise (Chellas, Brian F. 1980), apparently easy and in need of no hints, but in correspondence he was not sure what the solution was offhand. Segerberg passes over K (named differently) in his classic text as uninteresting. There are various reasons why K does indeed have limited applications in non-normal modal logics, but the ones that have been offered against K for monadic agency are fallacious in my opinion, an opinion shared by at least three others who have worked in this area, Risto Hilpinen, Andrew Jones and Mark Brown (in correspondence). Furthermore, the sort of problem noted with EK completeness proofs using standard neighborhood models, though solvable, appears to reiterate for other formulae, presumably *indefinitely*. Appendix 1 illustrates the problem of stalled completeness for system EK using standard neighborhood models and a preview of the approach to mending things back together used here. The strategy here taken is to modify the frames slightly and some of the semantic clauses associated with some formulae, and then have a uniform approach in all cases. The alternative of using a series of specifically tailored canonical models (as is done using the supplementation of a minimal canonical model for EKM completeness) for each such case is left for some other occasion. Although there are costs going in either direction, the approach within requires no on-the-fly gerrymandering to get completeness results. For an indication of the extent of work that might be needed to get completeness using standard neighborhood models and standard semantic clauses for just the cases of EK and EKC, see Van de Putte, Frederik, Paul McNamara et al. 2019.

extra parameter, *P,* which aids in some basic theorems entailing soundness results for all the consistent logics whose strong completeness was assured, and thus determination results for all such systems.

I will cast things with an eye toward generalization to classical modal logics and especially to various conditional logics (broadly conceived) for the dyadic systems. For this reason, in Part I, the "Monadic and/or Dyadic Agency" logics (MDAs) are defined weakly, although the intended interpretation that is nonetheless primarily in focus is that of logics of monadic and dyadic *agency* operators. Also, there are three sub-types of MDA logics considered: (Pure) Monadic Agency logics (PMAs), (Pure) Dyadic Agency logics (PDAs), and Dyadic-Monadic Agency logics (DMAs) containing both a monadic operator and a dyadic one.

In a planned extension, I hope to adapt and expand the results here to include monadic and dyadic ability operators and then amalgamations with the agency logics herein, converging on monadic-dyadic logics of agency and ability.[5]

Let me simply mention some contexts (especially but not exclusively, normative ones) where I think the logics have application when the agency interpretation is in mind. 1) Obviously, obligations often are to do things or make things happen.[6] So representing agential obligations calls for an agency operator. 2) The well-developed theory of *Normative Positions* utilizes a monadic agency operator.[7] However, there is good reason to think this can be extended and generalized to include dyadic agency in characterizing more fully one's agential normative position. For example, I may be obligated to bring about φ, but my normative position may be more fully characterized by considering further conditions I might meet (e.g. that I am permitted to bring about φ by bringing about ψ, forbidden to do so by bringing about ψ, obligated to do so thereby, etc. 3) Work on analysis of *Hohfeldian legal notions*: rights, privilege, powers … via (often directed) obligations might also be extended in analogous directions. For example, if I have a normative power over you to render you obligated to bring about φ, then that will be the sort of thing I can bring about only by bringing something else about, and often it will be of interest to know *by what means* I am able to do so or have done so. 4) Here are two special cases of the exercise of normative powers of interest in many normative systems. a) *Commitment* (in at least one sense) seems to involve dyadic agency (e.g. by promising to bring about φ, I render myself obligated to bring about φ); b) likewise for *consent* (e.g. by bringing it about that I agree to the surgery, I bring it about that it is permissible for the Doctor to perform it). 5) We must often reason about means to obligatory ends. If I'm obligated to bring about φ, then, unless bringing that about is a basic exercise of my agency, I will need to find some ψ such that I am able to bring about φ by bringing about ψ. 6) Dyadic agency seems crucial for understanding the important difference between basic and non-basic exercises of my agency. 7) More generally, for most any end that is not basic, we must reason about means to bringing about those ends, and so dyadic agency will be helpful here. 8) Obligation fulfillment is more fully specified if we specify how one fulfilled one's obligation: Jane fulfills her obligation to bring about φ by (her) bringing about ψ; or

[5] It appears that most of the ability additions can be generated by additions to the framework, and minor links to the agency operators (e.g. what Jane Doe brings about she is able to bring about), the importance of which is stressed rather persuasively in Governatori, Guido and Antonino Rotolo 2005.
[6] But not always—see McNamara, Paul 2004.
[7] See the references in footnote 3 and especially Sergot, Marek 2013.

Jane fulfills Jim's obligation to be such that his bill is paid by (Jane) bringing it about that Jim's bill is paid. 9) We are also interested in the normative status of fulfillments of obligations themselves, and special interest lies in *permissible fulfillment* (e.g. I'm obligated to bring it about that φ, I do so by bringing about ψ, *and* it is permissible to bring about φ by bringing about ψ). 10) We can also inquire into the status of an act of commitment itself: it may be beyond the call of duty for me to bring it about by volunteering to take on a dangerous mission that I thereby render myself obligated to do so.

In section 2, I specify three classes of logics, the pure monadic agency logics (PMAs), the pure dyadic agency logics (PDAs), and the Dyadic-Monadic Agency logics (DMAs), collectively the superclass of Monadic and Dyadic Agency logics (MDAs). (Appendix 2 briefly explores reductive schemes for the monadic and dyadic logics.) In section 3, I specify the variant neighborhood semantics for the MDAs with the extra parameter P and its modest algebraic structure, and prove that for any MDA model and formula in its associated language, the truth set for the formula on the model is contained in P. In section 4, I define the notion of a canonical model for any of the MDAs, show any canonical model is an MDA model, and with the aid of two minor lemmas, prove a fundamental theorem for canonical models, and strong completeness with respect to the frames of such models. In section 5, I prove twenty-five correspondence theorems to the effect that any canonical model for an MDA logic containing a given formula specified in section 2, satisfies an associated constraint specified in section 3. Section 6 briefly summarizes the correspondence theorems, and notes the derivative completeness results. Section 7 provides the key theorems for generating soundness results for all consistent MDA logics, and notes the derivative determination theorems for all the consistent MDAs. Finally, section 8 contains some brief concluding remarks.[8]

2 The PMA, PDA, DMA logics

We will be concerned with logics for languages with a set $PV = \{p_1, \ldots, p_n, \ldots\}$ of propositional variables and either or both of two agency operators:

BAφ: *Jane Doe brings it about that* φ
BA$'_\psi\varphi$: *Jane Doe brings it about that* φ *by bringing it about that* ψ.[9]

2.1 Monadic Agency

The monadic agency logic framework is as follows.

[8] Although much of the work in McNamara, Paul 2018 carries over, changes were called for in the frames of that earlier version in order to extend completeness results to determination results, which turned out to be not possible in the frames as previously specified. This required changes first and foremost in sections 3 and 4. Since I also consider more formulae (and so more logics) here, there are expansions in sections 2, 5, and 6. Section 7 with soundness and determination results is new. Lastly, there is a new sketch on applications in the introduction, and a second appendix on reductive schemes.

[9] Or *by bringing it about that* ψ *Jane Doe brings it about that* φ.

Formulae of Pure Monadic Agency (PMA) Logics:
1) All members of *PV* are PMA formulae.
2) The Propositional Constants, $\top$ and $\bot$, are PMA formulae.
3) If φ, ψ are PMA formulae, so are $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, and $\mathbf{B}\mathrm{A}\varphi$.

For generality, I define a weaker class of logics than those apt for agency alone:

A PMA Logic, *L*: *L* is a set of PMA formulae such that
1) All tautologous formulae are in *L*
2) *L* is closed under MP and closed under $\mathrm{RE}_{\mathbf{B}\mathrm{A}}$, if $\vdash \varphi \leftrightarrow \psi$, then $\vdash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}\psi$
3) *L* is non-trivial (does not contain all formulae).

We will consider the following further candidate axiom schemata for PMA logics, the first two of which are standard in most presentations of agency operators:

$\mathrm{T}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}\varphi \rightarrow \varphi$	[**B**A is an alethic operator]
$\mathrm{NO}_{\mathbf{B}\mathrm{A}}$:	$\neg\mathbf{B}\mathrm{A}\top$	[Nothing logically necessary is agential]
$\mathrm{C}_{\mathbf{B}\mathrm{A}}$:	$(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$	[(Agential) Composition/Conjunction][10]
$\mathrm{CS}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\neg\mathbf{B}\mathrm{A}\varphi \rightarrow \mathbf{B}\mathrm{A}\psi)$	[Conjunctive Syllogism][11]
$\mathrm{K}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}(\varphi \rightarrow \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \rightarrow \mathbf{B}\mathrm{A}\psi)$	[K principle][12]

[10] It is important to note that bringing about conjunctions in this tradition does not require bringing about both conjuncts, just making the conjunction true. For example, if φ is already settled true, then if one brings it about that ψ one thereby brings it about that the conjunction, $\varphi \wedge \psi$, is true. This is important for understanding the significance of the next overlooked and unusual modal scheme.

[11] $\mathrm{CS}_{\mathbf{B}\mathrm{A}}$ says that if I bring about the conjunction of two propositions, but not one of the conjuncts, then I bring about the other conjunct, or equivalently, if I bring about the conjunction, then I bring about at least one of the conjuncts (i.e. $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \vee \mathbf{B}\mathrm{A}\psi)$). It would seem that if I do bring about a conjunction, but not (say) its first conjunct, then that is because that conjunct is rendered true independently of my agency. But then the only way the truth of the conjunction could result from my agency is if the truth of the other conjunct results from it. Conversely, if I neither bring about p nor bring about q, then it would seem that I can't bring about both p and q. At the least, this formula seems worth exploring. CS is not considered (or validated) in Jones and Sergot 1996, Santos, Filipe, Andrew Jones et al. 1997, Santos and Carmo 1996, Elgesem 1997 or Elgesem 1993. I endorsed this principle in McNamara, Paul 2004. Note: should it be objected that one brings about a conjunction only if one brings about both conjuncts, that is, $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$, then CS is trivially true, but this is not the usual way of analyzing agential operators for conjunctions. Mark Brown reminded me of this possible objection in correspondence.

[12] K_{BA} is rejected by Walton and Elgesem (and not included in later writers in this tradition) via the following purported counterexample: *Green, by removing a platform, brings it about that if Brown falls from the roof of a certain building then Brown will die. Furthermore, Green brings it about that Brown falls from the roof of said building, say by pushing him off. Yet on the way down, Black shoots Brown dead. Thus Black, not Green, brings it about that Brown dies.* Note the shift in tense. This is not a counterexample to K, for as soon as Green pushes Brown at t in the scenario described, Green *does* bring it about at t that Brown *will* die (momentarily), and Black's shooting Brown on the way down does not change this prior fact. If we have Green push Brown at the exact moment that Black shoots Brown then it is plausible to say each brings it about that Brown will die (momentarily), in which case we have a classical instance of over-determination (cf. the killing of Caesar). So even if we adjust the times of acting to coincide, we should not then rule out by fiat over-determination of Brown's inevitable death, nor K thereby derivatively. See Elgesem, Dag 1993, p.83 for the above recasting of Walton's example, but see Walton, D 1975, pp. 105−110 for a much more extensive and nuanced discussion of agential K).

FS$_{\mathbf{B}\mathrm{A}}$ $(\varphi \wedge \mathbf{B}\mathrm{A}\psi) \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$ [Factual Supplementation][13]
M$_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$ [Decomposition][14]
R$_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$ [Composition-Decomposition].

2.2 Dyadic agency

We now turn to the dyadic systems. We also characterize the normal Pure Dyadic Agency logics (PDAs) very leanly with an eye to alternative interpretations of agency (e.g. where there is a time lag between the "antecedent" (means) and "consequent" (end)), but also with an eye to non-agential interpretations of the operator (e.g. as conditionals or propositionally-relativized modal operators), which may not involve some of the theses that are quite plausible for the instantaneous agency interpretation typical of the Kanger, Pörn, Elgesem tradition we have primarily in mind.

Formulae of Pure Dyadic Agency (PDA) Logics:
1) All members of *PV* are PMA formulae.
2) The Propositional Constants, $\top$ and $\bot$, are PMA formulae.
3) If φ, ψ are PDA formulae, so are: $\neg\varphi$, $(\varphi \wedge \psi)$, …, $(\varphi \leftrightarrow \psi)$ and $\mathbf{B}\mathrm{A}'_{\varphi}\psi$.

A PDA Logic, *L*: *L* is a set of PDA formulae such that:
0) *L* is non-trivial (does not contain all formulae)
1) All tautologous PDA formulae are in *L*
2) *L* is closed under MP
3) *L* is closed under RE$^{r}_{\mathbf{B}\mathrm{A}'}$ and RE$^{l}_{\mathbf{B}\mathrm{A}'}$: RE$^{r}_{\mathbf{B}\mathrm{A}'}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi$
RE$^{l}_{\mathbf{B}\mathrm{A}'}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\chi \leftrightarrow \mathbf{B}\mathrm{A}'_{\psi}\chi$.

We will consider the following candidate additional axiom schemata for PDA logics:

T$_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\psi}\varphi \rightarrow (\varphi \wedge \psi)$	[**B**A′ is an alethic operator]
NO$_{\mathbf{B}\mathrm{A}'}$:	$\neg(\mathbf{B}\mathrm{A}'_{\varphi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\varphi)$	[No Agency for Logical Truths]
AS$_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\varphi}\psi \rightarrow \neg\mathbf{B}\mathrm{A}'_{\psi}\varphi$	[Asymmetry]
IR$_{\mathbf{B}\mathrm{A}'}$:	$\neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi$	[Irreflexivity]
CC$_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}(\psi \wedge \chi)$	[Composition of "Consequents"]

[13] This is a controversial principle but a rationale might be offered to the effect that if φ is true by Jane's agency, then there is no problem, and if φ is true but not by her agency, then her agency is still involved in making the conjunction true, and specifically, she makes the conjunction true by bringing about ψ. (Cf. Humberstone, I. L. 2016, chapter 6.) There are various weakenings that might be expressed in a richer language, and perhaps found more plausible. For example, with agents and quantifiers, we could restrict φ to things not brought about by any other agent; alternatively, one could introduce a necessity operator so that φ would be restricted to things that are determined or in some sense necessary. However, these would also rule out multi-agent over-determination, but surely we want a logical framework which at least allows for some logics both compatible with and incompatible with over-determination.
[14] We will say a bit more about this often-included modal schema since it helps to reveal things about the primary intended interpretation. Although, M$_{\mathrm{BA}}$ might sound plausible at first blush, it is not in this agency tradition, and seeing why will reveal some nuances of this traditional approach to agency. Likewise for the next stronger schema.

$\mathrm{DC}_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi') \rightarrow \mathbf{B}\mathrm{A}'_{(\varphi \wedge \varphi')}(\psi \wedge \psi')$	[Double Composition][15]
$\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\neg\mathbf{B}\mathrm{A}'_{\chi}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi)$	[Dyadic Conjunctive Syllogism]
$\mathrm{CK}_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}(\varphi \rightarrow \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi)$	[Conditional K principle]
$\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\psi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\varphi) \rightarrow \mathbf{B}\mathrm{A}'_{(\psi \vee \chi)}\varphi$	[Disjunction of "Antecedents"][16]
$\mathrm{DD}_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi') \rightarrow \mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}(\psi \vee \psi')$	[Double Disjunction][17]
$\mathrm{TR}_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\psi}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\chi$	[Transitivity]
$\mathrm{CT}_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{(\varphi\wedge\psi)}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\chi$	[Cumulative Transitivity]
$\mathrm{SS}_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}\mathbf{B}\mathrm{A}'_{\varphi}\psi \rightarrow \mathbf{B}\mathrm{A}'_{(\chi \wedge \varphi)}\psi$	[Stage-Setting Principle][18]
$\mathrm{DFS}_{\mathbf{B}\mathrm{A}'}$:	$(\varphi \wedge \mathbf{B}\mathrm{A}_{\chi}\psi) \rightarrow \mathbf{B}\mathrm{A}_{\chi}(\varphi \wedge \psi)$	[Dyadic Factual Supplementation.]
$\mathrm{CM}_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}_{\chi}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}_{\chi}\psi)$	[Conditional Decomposition][19]
$\mathrm{CR}_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}_{\chi}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}_{\chi}\psi)$	[Conditional Composition-Decomposition]

2.3 Dyadic and monadic agency

The mixed dyadic and monadic agency logic framework is as follows.

Formulae of Dyadic-Monadic Agency (DMA) Logics:
1) All members of *PV* are PMA formulae.
2) The Propositional Constants, $\top$ and $\bot$ are PMA formulae.
3) If φ, ψ are DMA formulae, so are: $\neg\varphi$, $(\varphi \wedge \psi)$, …, $(\varphi \leftrightarrow \psi)$ and $\mathbf{B}\mathrm{A}\varphi$, $\mathbf{B}\mathrm{A}'_{\varphi}\psi$.

For the logics with both monadic and dyadic agency operators, one new schema will

[15] $\mathrm{CC}_{\mathbf{B}\mathrm{A}'}$ is a special case of this more general principle. Just let $\varphi = \varphi'$ and then apply $\mathrm{RE}^{l}_{\mathbf{B}\mathrm{A}'}$ to get $\mathrm{CC}_{\mathbf{B}\mathrm{A}'}$.
[16] This principle is unusual, but I think over-determination by the same agent is possible (e.g. I vote by raising each hand, ψ, χ); then the antecedent can be true where ψ and χ in the antecedent are not identical (not logically equivalent), and then the consequent plausibly holds—either of ψ and χ is sufficient; on the other hand, if such over-determination is not possible, then the principle is true because the only case where the antecedent can be true is where ψ and χ are identical (up to logical equivalence), and so the consequent would then hold trivially by $\mathrm{RE}^{l}_{\mathbf{B}\mathrm{A}'}$. In either event, it serves to introduce the analogs, for single agents, of the often discussed issue about multi-agent over-determination (e.g. the killing of Caesar).
[17] $\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$ is a special case of this more general principle. Just let $\psi = \psi'$ and then apply $\mathrm{RE}^{r}_{\mathbf{B}\mathrm{A}'}$ to get $\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$.
[18] This principle is unusual, but suppose as a chair of my department, following much discussion that appears to have ended, I request a show of hands by saying "All those in favor of the proposal—raise your hand?" (χ), and I do this while raising my own hand (φ). Then, by requesting this vote, I bring it about that by raising my hand, I bring it about that (ψ) I vote. This in turn implies that I bring it about that I vote by both saying what I said (χ) and raising my hand (φ). Neither of these acts alone would suffice for my voting: calling for a show of hands is what *sets the stage* for the possibility that raising my hand can constitute voting in favor. As with DA, perhaps the important thing is to raise the issue, in this case, of stage-setting agency. With an ability operator, and two agents, we can express my bringing it about by calling for a show of hands that you are able to vote by raising your hand. These sorts of agency cases are particularly important for changing normative positions by our actions in moral and institutional settings. Commitment for example can be analyzed along these lines with the addition of a personal (but not agential), obligation operator, **O**B (see McNamara, Paul 2004 on such an operator): for *Jane Doe is obligated to be such that* φ commits Jane Doe to ψ might be rendered as $\mathbf{B}\mathrm{A}'_{\varphi}\mathbf{O}\mathrm{B}\psi$, so that for example, Jane's bringing it about that she promises to meet you would ordinarily bring it about that she is obligated to bring it about that she does meet you. We can of course inquire into the status of this commitment itself.
[19] As with monadic $\mathrm{M}_{\mathbf{B}\mathrm{A}}$, this dyadic schema does not fit the intended agency framework well, and so a fortiori for the next schema. More below on these.

be central, one that says that if I bring about φ by bringing about ψ, then I bring about each:

S: $\mathbf{B}\mathrm{A}'_{\varphi}\psi \to (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$ [Separation].

For the combined system, for generality, we define the normal DMA logics as:

A DMA Logic, *L*: *L* is a set of DMA formulae such that:
0) *L* is non-trivial (does not contain all DMA formulae)
1) *L* contains all tautologous DMA formulae
2) *L* is closed under MP and $\mathrm{RE}_{\mathbf{B}\mathrm{A}}$ (see normal PMA logic above)
3) *L* is closed under $\mathrm{RE}^{r}_{\mathbf{B}\mathrm{A}'}$ and $\mathrm{RE}^{l}_{\mathbf{B}\mathrm{A}'}$ (see normal PDA logic above).

The base logics for the three logic types are as follows.

Base Pure Monadic Agency Logic (PMA):
SL: All Tautologies
MP: If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
$\mathrm{RE}_{\mathbf{B}\mathrm{A}}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}\psi$.

Base Pure Dyadic Agency Logic (PDA):
SL: All Tautologies
MP: If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
$\mathrm{RE}^{r}_{\mathbf{B}\mathrm{A}'}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi$ [Rule of Right Replacement for $\mathbf{B}\mathrm{A}'$]
$\mathrm{RE}^{l}_{\mathbf{B}\mathrm{A}'}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\chi \leftrightarrow \mathbf{B}\mathrm{A}'_{\psi}\chi$ [Rule of Left Replacement for $\mathbf{B}\mathrm{A}'$].

Base Dyadic-Monadic Agency System (DMA):
SL: All Tautologies
MP: If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
$\mathrm{RE}_{\mathbf{B}\mathrm{A}}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}\psi$
$\mathrm{RE}^{r}_{\mathbf{B}\mathrm{A}'}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi$
$\mathrm{RE}^{l}_{\mathbf{B}\mathrm{A}''}$: If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\chi \leftrightarrow \mathbf{B}\mathrm{A}'_{\psi}\chi$.

Let me note here that on the intended interpretation we might expect the following basic formulae to hold in all of what we might call the "*Preferred* DMA Logics":

$\mathrm{T}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}\varphi \to \varphi$
$\mathrm{NO}_{\mathbf{B}\mathrm{A}}$: $\neg\mathbf{B}\mathrm{A}\top$
$\mathrm{AS}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\psi}\varphi \to \neg\mathbf{B}\mathrm{A}'_{\varphi}\psi$ [Asymmetry]
S: $\mathbf{B}\mathrm{A}'_{\psi}\varphi \to (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$ [Separation].

All instances of $\mathrm{T}_{\mathbf{B}\mathrm{A}'}$ and $\mathrm{NO}'_{\mathbf{B}\mathrm{A}'}$ are theorems of all Preferred DMA logics:

$\mathrm{T}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\psi}\varphi \to (\varphi \wedge \psi)$

$\mathrm{NO}_{\mathbf{BA}'}$: $\neg(\mathbf{B}\mathrm{A}'_{\varphi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\varphi)$.[20]

Note also that Asymmetry entails Irreflexivity:

$\mathrm{IR}_{\mathbf{BA}'}$: $\neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi$ [Irreflexivity].[21]

Conversely, Irreflexivity combined with Transitivity entails Asymmetry.[22] Also it is obvious that $\mathbf{FS}_{\mathbf{B}\mathrm{A},}$ $(\varphi \wedge \mathbf{B}\mathrm{A}\psi) \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$, coupled with the T axiom, entails $\mathrm{C}_{\mathbf{BA}}$, $(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$.[23]

Let me also note here that modal principle M ($\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$) is not plausible for agency in this framework, however attractive at first blush. For given $\mathrm{RE}_{\mathbf{BA}}$ and $\mathrm{NO}_{\mathbf{BA}}$, it rules out all agential productivity:

If $\vdash \mathrm{M}_{\mathbf{BA}}$ and $\vdash \mathrm{NO}_{\mathbf{BA}}$, then $\vdash \neg\mathbf{B}\mathrm{A}\varphi$, for any φ.
Proof: Assume $\vdash \mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$ and suppose $\mathbf{B}\mathrm{A}\varphi$, for arbitrary φ. We have $\vdash \varphi \leftrightarrow (\varphi \wedge \top)$, so $\vdash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \top)$ by $\mathrm{RE}_{\mathbf{BA}}$, and then $\vdash \mathbf{B}\mathrm{A}\varphi \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \top)$, and so $\vdash \mathbf{B}\mathrm{A}\varphi \rightarrow \mathbf{B}\mathrm{A}\top$ from $\mathrm{M}_{\mathbf{BA}}$, and then finally, $\vdash \neg\mathbf{B}\mathrm{A}\varphi$ by $\mathrm{NO}_{\mathbf{BA}}$.

As one might expect, the dyadic version of M has a similar problem.[24] Of course, $\mathrm{R}_{\mathbf{BA}}$ and $\mathrm{R}_{\mathbf{BA}'}$ inherit these difficulties.[25]

3 Semantics for PMA, PDA, DMA logics

We first define the frames for the logics.

[20] For the first, Assume $\mathbf{B}\mathrm{A}'_{\psi}\varphi$. By S, $\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi$. So by $\mathrm{T}_{\mathbf{BA}}$, $\varphi \wedge \psi$. For the second, assume $\mathbf{B}\mathrm{A}'_{\varphi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\varphi$. So by S, $\mathbf{B}\mathrm{A}\top$, contrary to (monadic) $\mathrm{NO}_{\mathbf{BA}}$.

[21] For assume $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\psi \rightarrow \neg\mathbf{B}\mathrm{A}'_{\psi}\varphi$. So, in particular, $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\varphi \rightarrow \neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi$, and hence $\vdash \neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi$. I am well aware that certain efficiencies might be achieved by dropping irreflexivity (e.g. by then defining monadic agency via dyadic agency ($\mathbf{B}\mathrm{A}\varphi$ as $\mathbf{B}\mathrm{A}'_{\varphi}\varphi$)), and although this might be worth exploring technically (see Appendix 2 for more), I do not think irreflexivity (or asymmetry) can be denied given the intended notion of dyadic agency: that of *bringing about* φ *by bringing about* ψ. This notion, like that of "because", "in virtue of the fact that" and others, is intrinsically irreflexive. Put another way, if we allow for reflexive cases, we then must read $\mathbf{B}\mathrm{A}'_{\varphi}\psi$ along some such lines as *Either Jane Doe brings it about that* ψ *by bringing it about that* φ, *or, Jane brings it about that* ψ *and Jane brings it about that* φ, the latter now being equipped to plausibly handle $\mathbf{B}\mathrm{A}'_{\varphi}\varphi$.

[22] For reductio, assume $\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\psi}\varphi$. Then Transitivity gives us $\mathbf{B}\mathrm{A}'_{\varphi}\varphi$, contrary to Irreflexivity.

[23] See Humberstone, I. L. 2016, p.451.

[24] We show that if $\vdash \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi)$ and $\vdash \neg(\mathbf{B}\mathrm{A}'_{\chi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\chi)$, then $\vdash \neg\mathbf{B}\mathrm{A}'_{\chi}\varphi$, for any φ.
Proof:
Assume $\vdash \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi)$ and suppose $\mathbf{B}\mathrm{A}'_{\chi}\varphi$, for arbitrary φ *and* χ. We have $\vdash \varphi \leftrightarrow (\varphi \wedge \top)$, so $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \top)$ by $\mathrm{RE}^{\mathrm{r}}_{\mathbf{BA}'}$, and then $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \top)$, and so $\vdash \mathbf{B}\mathrm{A}'_{\chi}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{\chi}\top$ from $\mathrm{M}_{\mathbf{BA}'}$, and then finally, $\vdash \neg\mathbf{B}\mathrm{A}'_{\chi}\varphi$ by $\mathrm{NO}_{\mathbf{BA}'}$.

[25] I regret that I have not been able to explore interactions between some of these schemata much here. For example, one anonymous reviewer pointed out a more subtle entailment: irreflexivity coupled with double composition (of consequents) entails asymmetry. Suppose for reductio that $\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\psi}\varphi$ for some φ and ψ. Then composition of consequents yields $\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\psi \wedge \varphi)$, and by one application of RE (right or left) we have a violation of irreflexivity, and so asymmetry follows.

A preliminary definition will be convenient, then the frame definitions:

A set P of subsets of W is *proper* iff (if and only if) it meets these closure conditions:
($P1$) P is closed under complementation regarding W: if $X \in P$, then $W–X \in P$;
($P2$) P is closed under unions: if X, $Y \in P$ then $X \cup Y \in P$.[26]

A *PMA Frame*, $F = <W, P, f^1>$:
1) W is non-empty [Worlds]
2) $P \subseteq \mathrm{Pow}(W)$ [The Propositions][27]
3) f^1: $W \rightarrow \mathrm{Pow}(P)$ [Maps worlds to sets of propositions]
4) P is a proper set of subsets of W meeting this additional condition:
($P3$) P is closed under b: if $X \in P$, then $b(X) \in P$, where $b(X) = \{w \in W: X \in f^1(w)\}$.[28]

A *PDA Frame*, $F = <W, P, f^2>$:
1) W is non-empty [Worlds]
2) $P \subseteq \mathrm{Pow}(W)$ [The Propositions]
3) f^2: $W \times P \rightarrow \mathrm{Pow}(P)$ [Maps world-proposition pairs to sets of propositions]
4) P is a proper set of subsets of W meeting this additional condition:
($P4$) P is closed under b': if $X \in P$ & $Y \in P$, then $b'(Y, X) \in P$, where $b'(Y, X) = \{w \in W: Y \in f^2(w, X)\}$.[29]

A *DMA Frame*, $F = <W, P, f^1, f^2>$:
1) W is non-empty
2) $P \subseteq \mathrm{Pow}(W)$
3) f^1:$W \rightarrow \mathrm{Pow}(P)$
4) f^2: $W \times P \rightarrow \mathrm{Pow}(P)$
5) P is a proper set of subsets of W meeting these additional conditions:
($P3$): P is closed under b
($P4$): P is closed under b'.

Additional normal PMA and DMA frames will be considered with one or more of these clauses for the monadic operator:

t) If $X \in f^1(w)$, then $w \in X$
no) $W \notin f^1(w)$
c) If $X \in f^1(w)$ & $Y \in f^1(w)$, then $X \cap Y \in f^1(w)$

---

26 If P is a non-empty proper set of subsets of W, it follows immediately that $\varnothing$ and W are elements of P. It will turn out that in any DMA frame, $P \neq \varnothing$. Closure under intersection follows for $X \cap Y = (X' \cup Y')'$.
27 Or the admissible/expressible propositions: P can be any subset of Pow(W), consistent with constraints to be placed on P resulting in its being the case that the truth sets of all formulae in all models are in P. P will be used to facilitate correspondence proofs for completeness that otherwise stall as illustrated in Appendix 1; and imposing modest structure on P will facilitate validity proofs for soundness that would otherwise fail.
28 Intuitively, $b(X)$ is the set (perhaps empty) of worlds where the proposition X is brought about.
29 Intuitively, $b'(X, Y)$ is the set of worlds (perhaps empty) where the proposition Y is brought about by bringing about X.

cs) <u>If $X \in P$, $Y \in P$</u>, then if $X \cap Y \in f^1(w)$ then $X \in f^1(w)$ or $Y \in f^1(w)$[30]
k′) <u>If $Y \in P$</u>, $-X \cup Y \in f^1(w)$ and $X \in f^1(w)$, then $Y \in f^1(w)$
s) <u>If $X \in P$</u> & $w \in X$ & $Y \in f^1(w)$, then $X \cap Y \in f^1(w)$
m′) <u>If $X, Y \in P$</u> & $X \cap Y \in f^1(w)$, then $X \in f^1(w)$[31]
r) <u>If $X, Y \in P$</u>, then $X \cap Y \in f^1(w)$ iff $X \in f^1(w)$ & $Y \in f^1(w)$.

Additional normal PDA and DMA frames will be considered with one or more of these clauses for the dyadic operator:

t′) If $Y \in f^2(w, X)$, then $w \in Y \cap X$
no′) $\neg\exists X(W \in f^2(w, X))$ and $f^2(w, W) = \varnothing$
as) If $Y \in f^2(w, X)$, then $X \notin f^2(w, Y)$
ir) $X \notin f^2(w, X)$
cc) If $Y \in f^2(w, X)$ and $Z \in f^2(w, X)$, then $Y \cap Z \in f^2(w, X)$
dc) If $Y \in f^2(w, X)$ and $Y' \in f^2(w, X')$, then $Y \cap Y' \in f^2(w, X \cap X')$
dcs) <u>If $X \in P$, $Y \in P$</u> and $X \cap Y \in f^2(w, Z)$ then $X \in f^2(w, Z)$ or $Y \in f^1(w, Z)$
ck′) <u>If $Y \in P$</u>, $-X \cup Y \in f^2(w, Z)$, and $X \in f^2(w, Z)$, then $Y \in f^2(w, Z)$
da) If $X \in f^2(w, Y)$ and $X \in f^2(w, Z)$, then $X \in f^2(w, Y \cup Z)$
dd) If $Y \in f^2(w, X)$ and $Y' \in f^2(w, X')$, then $Y \cup Y' \in f^2(w, X \cup X')$
tr) If $Y \in f^2(w, X)$ and $Z \in f^2(w, Y)$, then $Z \in f^2(w, X)$
ct) If $Y \in f^2(w, X)$ and $Z \in f^2(w, X \cap Y)$, then $Z \in f^2(w, X)$
ss) <u>If $X \in P$, $Y \in P$</u>, $Z \in f^2(w, U)$ and $Z = \{w' : Y \in f^2(w', X)\}$, then
$Y \in f^2(w, U \cap X)$
dfs) <u>If $X \in P$</u> & $w \in X$ & $Y \in f^2(w, Z)$, then $X \cap Y \in f^2(w, Z)$
c*m′*) <u>If $X, Y \in P$</u> & $X \cap Y \in f^2(w, Z)$, then $X \in f^2(w, Z)$
cr) <u>If $X, Y \in P$</u>, then $X \cap Y \in f^2(w, Z)$ iff $X \in f^2(w, Z)$ & $Y \in f^2(w, Z)$.

We will also consider this key clause linking f^1 and f^2,

s) If $Y \in f^2(w, X)$, then $X \in f^1(w)$ & $Y \in f^1(w)$. [Dyadic-Monadic Bridge].

However, for DMA frames with clauses ***t***, ***no***, ***ir***, and ***s***, clauses ***t′*** and ***no′*** are derivable, just as their sentential analogs, $\mathrm{T}_{\mathbf{BA}'}$ and $\mathrm{NO}_{\mathbf{BA}'}$, are derivable in the DMA logics containing $\mathrm{T}_{\mathbf{BA}}$, $\mathrm{NO}_{\mathbf{BA}}$, $\mathrm{IR}_{\mathbf{BA}'}$, and S.

[30] Note the relativization to P in the antecedent for this constraint, and others below. I will use such <u>underlining</u> to remind the reader of those places where it appears that a straightforward correspondence proof would stall (as illustrated in Appendix 1) and so where such relativization to P is invoked and facilitates proving the correspondence theorems for the associated formulae in the canonical models as defined (without tinkering), and thus to getting completeness. In all other cases, no relativization to P is invoked or useful. More on this below.
[31] The Constraint ***m′*** is equivalent to <u>if $X, Y \in P$</u> & $X \subseteq Y$ and $X \in f^1(w)$, then $Y \in f^1(w)$, as well as the more explicit form, <u>if $X, Y \in P$</u> & $X \cap Y \in f^1(w)$, then $X \in f^1(w)$ *and* $Y \in f^1(w)$. If we drop the relativization to P we appear to run into the same problem as indicated in Appendix 1. We use parameter P to resolve this, whereas others use supplementation of the models. P relativization appears to provide one systematic way to prove that any canonical model for an MDA logic meets the associated P-relativized constraint.

A *PMA Model*, $M = \langle F, V\rangle$, where F is an PMA frame, $\langle W, P, f^1\rangle$ such that V: $PV \rightarrow P$, where V maps the propositional variables to elements of P, the expressible propositions.

Similarly, for a PDA Model and for a DMA Model.

Truth on a PMA Model, $M = \langle\langle W, P, f^1\rangle, V\rangle$:

[PV] If $\varphi \in PV$, $M, w \vDash \varphi$ iff $w \in V(\varphi)$

[$\top$] $M, w \vDash \top$, for each $w \in W$

[$\bot$] $M, w \nvDash \bot$, for each $w \in W$

[$\neg$] $M, w \vDash \neg\varphi$ iff $M, w \nvDash \varphi$

[$\wedge$] $M, w \vDash (\psi \wedge \chi)$ iff $M, w \vDash \psi$ and $M, w \vDash \chi$

. . .

[$\rightarrow$] $M, w \vDash (\psi \rightarrow \chi)$ iff $M, w \nvDash \psi$ or $M, w \vDash \chi$

[**B**A] $M, w \vDash \mathbf{B}\mathrm{A}\psi$ iff $[\![\psi]\!]^M \in f^1(w)$, where $[\![\psi]\!]^M$ is $\{w\text{: } M, w \vDash \psi\}$, "the truth set of φ for M".

Truth on an PDA Model, M: As above, but replace f^1 with f^2 and [**B**A] with:

[**B**A′] $M, w \vDash \mathbf{B}\mathrm{A}'_{\varphi}\psi$ iff $[\![\psi]\!]^M \in f^2(w,[\![\varphi]\!]^M)$.

Truth on an DMA Model, $M = \langle\langle W, P, f^1, f^2\rangle, V\rangle$: Same clauses as with PMA, but add clause [**B**A′] to the truth conditions.

We now show that the truth sets for all formulae are in P for any of the MDA logics and associated models. This will facilitate soundness proofs, and thus be of aid in determination proofs.

Lemma 1:

For any MDA model M, and formula φ in the associated language, $[\![\varphi]\!]^M \in P$.

Proof: Induction is on the complexity of the formulae.[32]

Recall that by definition $[\![\varphi]\!]^M = \{w\text{: } M, w \vDash \varphi\}$.

Base Case: φ is a member of PV. By definition of V in M, $V(\varphi) \in P$, and by [PV] in the truth clauses, it follows that $V(\varphi) = [\![\varphi]\!]^M$.

Inductive Case:

a) Suppose $\varphi = \neg\psi$, for some ψ. By inductive hypothesis (IH), $[\![\psi]\!]^M \in P$. So by *P1*, $W - [\![\psi]\!]^M \in P$, that is, $[\![\neg\psi]\!]^M \in P$.

b) Suppose $\varphi = (\psi \vee \chi)$, for some ψ and χ. By IH, $[\![\psi]\!]^M \in P$ and $[\![\chi]\!]^M \in P$. So by *P2*, $[\![\psi]\!]^M \cup [\![\chi]\!]^M \in P$, and so $[\![\psi \vee \chi]\!]^M \in P$.

c) Suppose $\varphi = \mathbf{B}\mathrm{A}\psi$, for some ψ. By IH, $[\![\psi]\!]^M \in P$. By clause *P3*, $b[\![\psi]\!]^M \in P$, so by definition of b, $\{w \in W\text{: } [\![\psi]\!]^M \in f^1(w)\} \in P$. But then by [**B**A], it follows that $\{w \in W\text{: } M, w \vDash \mathbf{B}\mathrm{A}\psi\} \in P$, that is, $[\![\mathbf{B}\mathrm{A}\psi]\!]^M \in P$.

d) Suppose $\varphi = \mathbf{B}\mathrm{A}'_{\chi}\psi$, for some ψ and χ. By IH, $[\![\psi]\!]^M \in P$ and $[\![\chi]\!]^M \in P$. By clause *P4*, $b'([\![\psi]\!]^M, [\![\chi]\!]^M) \in P$, so by definition of b', $\{w \in W\text{: } [\![\psi]\!]^M \in f^2([\![\chi]\!]^M, w)\} \in P$.

[32] Given standard truth functional equivalences, it suffices to cover atomics, negation, disjunction (and then the modal operators).

But then by [$\mathbf{B}$A′], it follows that $\{w \in W\colon M, w \models \mathbf{B}\mathrm{A}'_{\chi}\psi\} \in P$, that is, $[\![\mathbf{B}\mathrm{A}'_{\chi}\psi]\!]^M \in P$.

4 The Fundamental Theorem for Canonical Models

We now define the canonical models for the MDA logics (i.e. the PMA, PDA, and DMA logics), and prove a fundamental theorem for such models, and then proceed to prove various theorems linking logics containing all instances of the preceding formulae schemata we listed as theorems and their canonical models.[33] In a familiar way, these will entail a large array of completeness results.

Let Σ^L be the set of maximal consistent sets (MCSs) of formulae for MDA logic, L. Then let $|\varphi|^L$ be the set of MCSs that contain φ: $\{\delta \in \Sigma^L\colon \varphi \in \delta\}$.[34]

For any given MDA logic, L, from here on, the superscript "L" will be left as understood for a canonical model of L and its components (unless invoked for emphasis/clarity).

A *Canonical Model*, $M = \langle W, P, f^1, f^2, V\rangle$ for any DMA logic, L, is defined as follows:

a) $W = \Sigma$

b) $P = \{X\colon \exists\varphi(|\varphi| = X)\}$

c) $f^1(w) = \{X\colon \exists\varphi(|\varphi| = X \;\&\; \mathbf{B}\mathrm{A}\varphi \in w)\}$
(So $X \in f^1(w)$ iff there is a formula, $\mathbf{B}\mathrm{A}\varphi$, in w such that X is the set of the MCSs containing φ.)[35]

d) $f^2(w, X) = \{Y\colon \exists\psi(|\psi| = Y \;\&\; \exists\varphi(|\varphi| = X \;\&\; \mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))\}$
(So $Y \in f^2(w, X)$ iff there is a formula $\mathbf{B}\mathrm{A}'_{\varphi}\psi$ in w such that Y is the set of MCSs containing ψ and X is the set of MCSs containing φ.)[36]

e) $V(Pn) = |Pn|$.

Canonical Models for PMA logics: drop f^2 and *d*.
Canonical Models for PDA logics: drop f^1 and *c*.

We note these *Basic Properties* (BP) *of MCSs* apply to W in our canonical model:

$\vdash \varphi$ iff $\forall w \in W\colon \varphi \in w$;
$|\neg\varphi| = \Sigma - |\varphi|$;
$|\varphi \wedge \psi| = |\varphi| \cap |\psi|$;
$|\varphi \vee \psi| = |\varphi| \cup |\psi|$;
$\vdash \varphi \leftrightarrow \psi$ iff $|\varphi| = |\psi|$.

The following two-part lemma will come in handy.

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[33] We will often talk of a logic as containing a theorem (e.g. "containing theorem $\mathrm{T}_{\mathbf{BA}}$") where this will be understood as shorthand for a logic containing all instances of the schemata in question.

[34] See for example Chapter 2.6 of Chellas, Brian F. 1980.

[35] Given $\mathrm{RE}_{\mathbf{BA}}$, it will turn out that $|\varphi| \in f^1(w)$ iff $\mathbf{B}\mathrm{A}\varphi \in w$. See Lemma 2*a*.

[36] Given $\mathrm{RE}^{\mathrm{r}}_{\mathbf{BA}'}$ and $\mathrm{RE}^{\mathrm{l}}_{\mathbf{BA}'}$, it will turn out that $|\psi| \in f^2(w, |\varphi|)$ iff $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$. See Lemma 2*b*.

Lemma 2:
a) For any φ and w: $\mathbf{B}\mathrm{A}\varphi \in w$ iff $|\varphi| \in f^1(w)$.
Proof:
$\Rightarrow$: Suppose $\mathbf{B}\mathrm{A}\varphi \in w$. Then since $|\varphi| = |\varphi|$, we have $\exists\varphi(|\varphi| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}\varphi \in w)$, which suffices by clause *c* for $|\varphi| \in f^1(w)$.
$\Leftarrow$: Assume $|\varphi| \in f^1(w)$. So by clause *c*, $\exists\varphi'(|\varphi'| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}\varphi' \in w)$. So fixing φ', we have $|\varphi'| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}\varphi' \in w$. So by BP, $\vdash \varphi \leftrightarrow \varphi'$, and then by $\mathrm{RE}_{\mathbf{B}\mathrm{A}}$, $(\mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}\varphi') \in w$, and so $\mathbf{B}\mathrm{A}\varphi \in w$.
b) For any φ, ψ, and w: $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$ iff $|\psi| \in f^2(w, |\varphi|)$.
Proof:
$\Rightarrow$: Suppose $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$. Then since $|\psi| = |\psi|$ and $|\varphi| = |\varphi|$, we have $\exists\psi(|\psi| = |\psi|\ \&$ $\exists\varphi(|\varphi| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}'_\varphi\psi \in w))$, which suffices by clause *d* for $|\psi| \in f^2(w, |\varphi|)$.
$\Leftarrow$: Assume $|\psi| \in f^2(w, |\varphi|)$. So by clause *d*, $\exists\psi'(|\psi'| = |\psi|\ \&\ \exists\varphi'(|\varphi'| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. So fixing ψ' and φ', we have $|\psi'| = |\psi|\ \&\ |\varphi'| = |\varphi|\ \&\ \mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. So by BP, $\vdash \varphi \leftrightarrow \varphi'$ and $\vdash \psi \leftrightarrow \psi'$, and hence by $\mathrm{RE}^{\mathrm{r}}_{\mathbf{B}\mathrm{A}'}$ and $\mathrm{RE}^{\mathrm{l}}_{\mathbf{B}\mathrm{A}'}$, we get $(\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \leftrightarrow \mathbf{B}\mathrm{A}'_\varphi\psi) \in w$, and so $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$.

We now show that the canonical model of any MDA logic as defined above is really an MDA model. This will be needed in using our correspondence theorems in section 5. In the proof below, it is straightforward to separate the components for the three types of logics. For PMA logics, only clauses 1–3 and 5 sub-clauses *P1* through *P3* below are relevant; for PDA logics only clauses 1, 2, 4 and 5 sub-clauses *P1*, *P2*, and *P4* below are relevant, and for DMA logics, all clauses below are relevant.

***Lemma 3*:**
For any canonical model, M^L, for an MDA logic, L, M^L is an MDA model.
Proof:
1) $W \neq \varnothing$. Since by definition, no MDA logic contains all MDA formulae, some formula φ will be a non-theorem, so there will be a maximal consistent extension of $\neg\varphi$, contained in W.
2) $P \subseteq \mathrm{Pow}(W)$, for by definition, for each $X \in P$, there is a formula φ such that $|\varphi| = X$, and by design of W, $|\varphi| \subseteq W$.
3) f^1: $W \rightarrow \mathrm{Pow}(P)$, for by definition of f^1, its domain is W, and for any such w, $f^1(w)$ is $\{X: \exists\varphi(|\varphi| = X\ \&\ \mathbf{B}\mathrm{A}\varphi \in w\}$, but by definition of P, it must contain all such Xs.
4) f^2: $W \times P \rightarrow \mathrm{Pow}(P)$, for by definition of f^2, its domain must be $W \times P$, for its Image for any pair (w, X) in its domain is $\{Y: \exists\psi(|\psi| = Y\ \&\ \exists\varphi(|\varphi| = X\ \&\ \mathbf{B}\mathrm{A}'_\varphi\psi \in w))\}$, and by definition of P, it must contain all such values of X and Y.
5) Recall that by definition of the canonical model, $P = \{|\varphi|: \varphi$ is a formulae of the logic, $L\}$. P is clearly a proper subset of W and we show it meets conditions *P1* – *P4* as well:
   (*P1*) P is closed under complementation regarding W: If $|\varphi| \in P$, for some φ, then by definition of P (in M^L), $|\neg\varphi| \in P$. But by BP, $|\neg\varphi| = \Sigma - |\varphi|$. So $W - |\varphi| \in P$.
   (*P2*) P is closed under unions: Suppose for some φ and ψ, $|\varphi| \in P$ and $|\psi| \in P$. Then by definition of P, $|\varphi \vee \psi| \in P$; but by BP, $|\varphi \vee \psi| = |\varphi| \cup |\psi|$, so $|\varphi| \cup |\psi| \in P$.

($P3$) P is closed under b: if $|\varphi| \in P$, then $|\mathbf{B}A\varphi| \in P$, that is $\{w \in W: \mathbf{B}A\varphi \in w\} \in P$. By Lemma $2a$, $\{w \in W: \mathbf{B}A\varphi \in w\} = \{w \in W: |\varphi| \in f^1(w)\}$. So $\{w \in W: |\varphi| \in f^1(w)\} \in P$, that is, $b(|\varphi|) \in P$.
($P4$) P is closed under b': if $|\varphi| \in P$ & $|\psi| \in P$, then $|\mathbf{B}A'_\varphi\psi| \in P$, that is, $\{w \in W: \mathbf{B}A'_\varphi\psi \in w\} \in P$. By Lemma $2b$, $\{w \in W: \mathbf{B}A'_\varphi\psi \in w\} = \{w \in W: |\psi| \in f^2(w, |\varphi|)\}$. So $\{w \in W: |\psi| \in f^2(w, |\varphi|)\} \in P$, that is, $b'(|\psi|, |\varphi|) \in P$.

We can now easily prove the fundamental theorem:

***(FT) Fundamental Theorem for the Canonical Models for MDA Logics*:**

$M, w \vDash \varphi$ iff $\varphi \in w$, that is, $[\![\varphi]\!]^M = |\varphi|$.

Proof: Assume the theorem is to be proved in the usual way by induction on the complexity of the formulae, and that it is already proved for formulae whose main connective is one of our truth-functional operators. (The base case holds by stipulation of clause e of the definition of a canonical model for any MDA logic.) We show that it holds for the remaining possible formula types, $\mathbf{B}A\psi$, and $\mathbf{B}A'_\chi\psi$.
A) Suppose $\varphi = \mathbf{B}A\psi$, for some ψ. So by IH, for every w, $M, w \vDash \psi$ iff $\psi \in w$, thus $[\![\psi]\!]^M = |\psi|$. By the semantic clause for $\mathbf{B}A\psi$, $M, w \vDash \mathbf{B}A\psi$ iff $[\![\psi]\!]^M \in f^1(w)$. So $M, w \vDash \mathbf{B}A\psi$ iff $|\psi| \in f^1(w)$. But by lemma $2a$, $|\psi| \in f^1(w)$ iff $\mathbf{B}A\psi \in w$. Thus $M, w \vDash \mathbf{B}A\psi$ iff $\mathbf{B}A\psi \in w$.
B) Suppose $\varphi = \mathbf{B}A'_\chi\psi$, for some χ, ψ. So by IH, $[\![\chi]\!]^M = |\chi|$ and $[\![\psi]\!]^M = |\psi|$. By the semantics clause for $\mathbf{B}A'_\chi\psi$, $M, w \vDash \mathbf{B}A'_\chi\psi$ iff $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. So $M, w \vDash \mathbf{B}A'_\chi\psi$ iff $|\psi| \in f^2(w, |\chi|)$. But by lemma $2b$, $\mathbf{B}A'_\chi\psi \in w$ iff $|\psi| \in f^2(w, |\chi|)$. So $M, w \vDash \mathbf{B}A'_\chi\psi$ iff $\mathbf{B}A'_\chi\psi \in w$.

Thus the theorem holds generally, and so the theorems of any MDA logic, L, are exactly those valid in any canonical model, M^L, with case A pertaining to the PMA logics, and case B pertaining to the PDA logics and both pertaining to the DMA logics.

Strong Completeness Corollary of FT: For any canonical model $M = \langle W, P, f^1, f^2, V\rangle$ for an MDA logic L, where F is the frame $\langle W, P, f^1, f^2\rangle$ of that model: if $\Gamma \vDash_F \varphi$, then $\Gamma \vdash_L \varphi$ (where Γ is any set of formula for the language of L).
Proof: Suppose $\Gamma \nvdash_L \varphi$. Then $\Gamma \cup \neg\varphi$ is an L-consistent set. L contains classical propositional logic, so by Lindenbaum's Lemma, for some maximal L-consistent set, w, $\Gamma \cup \neg\varphi \subseteq w$. So by the Fundamental Theorem, there is a canonical model M with frame F such that $\forall \psi \in \Gamma$, $M, w \vDash \psi$, and $M, w \nvDash \varphi$, so $\Gamma \nvDash_F \varphi$.

With these proofs in place, we focus on correspondences between the key formulae we have listed above and the frame constraints we have informally associated with them, also listed above (e.g. $T_{\mathbf{B}A}$ with ***t***). For each such association, we will show in the next section that if any MDA logic contains one of these formulae, then any canonical model for that logic must satisfy the associated constraint. Given our fundamental theorem, these correspondence proofs will imply that for any (consistent) MDA logic, L, specified by any (consistent) combination of the formulae below, any of its canonical models will satisfy the combination of associated constraints. This in turn will entail a strong completeness theorem for any such L with respect to any class of models that contains a canonical model for L, and given the Corollary of FT just above, it will follow that the logic is strongly complete for the intended class of frames.

5 Correspondence results for completeness

For theorems where the proof depends on the propositions involved being relativized to para-meter P in the structures, we will label the theorem number with a superscript "p" (e.g. see T4p).[37]

T1. Any canonical model for an MDA logic with $T_{\mathbf{B}A}$ satisfies the constraint ***t***: If $X \in f^1(w)$, then $w \in X$. Suppose $X \in f^1(w)$. So $\exists\varphi(|\varphi| = X \ \&\ \mathbf{B}A\varphi \in w)$. Fixing φ, it follows that $|\varphi| = X$ and $\mathbf{B}A\varphi \in w$. But since $\vdash \mathbf{B}A\varphi \rightarrow \varphi$, $\mathbf{B}A\varphi \rightarrow \varphi \in w$, and thus $\varphi \in w$. Thus, $w \in |\varphi|$, that is $w \in X$.

T2. Any canonical model for an MDA logic with $NO_{\mathbf{B}A}$ satisfies the constraint ***no***: $W \notin f^1(w)$. Assume that $W \in f^1(w)$. So $\exists\varphi(|\varphi| = W$ and $\mathbf{B}A\varphi \in w)$. Fixing φ, we have $|\varphi| = W$ and $\mathbf{B}A\varphi \in w$. But $|\varphi| = W$ iff $\vdash \varphi \leftrightarrow \top$, so by $RE_{\mathbf{B}A}$, $\vdash \mathbf{B}A\varphi \leftrightarrow \mathbf{B}A\top$, and so $\mathbf{B}A\top \in w$. Yet $\vdash \neg\mathbf{B}A\top$, so $\neg\mathbf{B}A\top \in w$, contrary to assumption that w is consistent.

T3. Any canonical model for an MDA logic with $C_{\mathbf{B}A}$ satisfies the constraint ***c***: If $X \in f^1(w)$ and $Y \in f^1(w)$, then $X \cap Y \in f^1(w)$. Suppose $X \in f^1(w)$ and $Y \in f^1(w)$. So by the reasoning above in T1, we have that there exists a φ and a ψ such that: $|\varphi| = X$ and $|\psi| = Y$ and $\mathbf{B}A\varphi \in w$ and $\mathbf{B}A\psi \in w$. But $\vdash (\mathbf{B}A\varphi \wedge \mathbf{B}A\,\psi) \rightarrow \mathbf{B}A(\varphi \wedge \psi)$, so $\mathbf{B}A(\varphi \wedge \psi) \in w$. But $|\varphi \wedge \psi| = |\varphi \wedge \psi|$, so $\exists Z(Z = |\varphi \wedge \psi| \ \&\ \mathbf{B}A(\varphi \wedge \psi) \in w)$, and so $|\varphi \wedge \psi| \in f^1(w)$. But $|\varphi \wedge \psi| = |\varphi| \cap |\psi| = X \cap Y$, so $X \cap Y \in f^1(w)$.

The next five correspondence proofs utilize the parameter, P, and the next two footnotes explain P's utility.

T4^P. Any canonical model for an MDA logic with $CS_{\mathbf{B}A}$ satisfies the constraint ***cs***: If $\underline{X, Y \in P}$ and $X \cap Y \in f^1(w)$, then $X \in f^1(w)$ or $Y \in f^1(w)$. Suppose for *reductio* that (1) $X, Y \in P$, (2) $X \cap Y \in f^1(w)$, and (3) $X \notin f^1(w)$ and (4) $Y \notin f^1(w)$. So from 2 we have $\exists\varphi(|\varphi| = X \cap Y \ \&\ \mathbf{B}A\varphi \in w)$, and then fixing φ, we get (2′) $|\varphi| = X \cap Y \ \&\ \mathbf{B}A\varphi \in w$. Given 3 and 4 it follows that $\neg\exists\psi(|\psi| = X \ \&\ \mathbf{B}A\psi \in w)$ and $\neg\exists\chi(|\chi| = Y \ \&\ \mathbf{B}A\chi \in w)$, that is (3′) $\forall\psi$(if $|\psi| = X$, then $\mathbf{B}A\psi \notin w$) and (4′) $\forall\chi$(if $|\chi| = Y$ then $\mathbf{B}A\chi \notin w$). But given 1, it follows from clause b of M that $\exists\psi(|\psi| = X)$ and $\exists\chi(|\chi| = Y)$, and then instantiating, we get $|\psi| = X$ and $|\chi| = Y$.[38] Then from 3′ and 4′, we get $\mathbf{B}A\psi \notin w$ and $\mathbf{B}A\chi \notin w$. Given 2 and the identifications above for X, Y, we have $|\psi| \cap |\chi| \in f^1(w)$, and by BP, we get $|\psi \wedge \chi| \in f^1(w)$. Then from lemma 2$a$, it follows that $\mathbf{B}A(\psi \wedge \chi) \in w$. So

[37] As well as continuing to use underlining as explained in note 31.
[38] Here is our first case other than K where we encounter the same sort of problem illustrated in Appendix 1 for K: How do we know that Y is expressible? (See T5^P and the next note on the case of K itself.)

we have $\mathbf{B}\mathrm{A}(\psi \wedge \chi) \in w$, $\mathbf{B}\mathrm{A}\psi \notin w$, and $\mathbf{B}\mathrm{A}\chi \notin w$, but $\mathrm{CS}_{\mathbf{BA}}$ is also in w, so w turns out to be not an MCS.

T5^P. Any canonical model for an MDA logic with $\mathrm{K}_{\mathbf{BA}}$ satisfies constraint ***k'***, that if $Y \in P$, $-X \cup Y \in f^1(w)$, and $X \in f^1(w)$, then $Y \in f^1(w)$. Assume (1) $Y \in P$, and (2) $-X \cup Y \in f^1(w)$ and $X \in f^1(w)$. From 2 by clause c of M, $\exists\varphi(|\varphi| = -X \cup Y$ & $\mathbf{B}\mathrm{A}\varphi \in w)$ and $\exists\psi(|\psi| = X$ & $\mathbf{B}\mathrm{A}\psi \in w)$. Fixing φ and ψ, we get (2′) $|\varphi| = -X \cup Y$ & $\mathbf{B}\mathrm{A}\varphi \in w$ and $|\psi| = X$ & $\mathbf{B}\mathrm{A}\psi \in w$. Since $|\psi| = X$, $-|\psi| = -X$, and so by BP, $|\neg\psi| = -X$. From assumption 1, by clause b of M, we get $\exists\chi(|\chi| = Y)$.[39] Fixing χ, we have $|\chi| = Y$. Substituting for $-X$ and Y in the first conjunct of 2′, we get $|\varphi| = |\neg\psi| \cup |\chi|$. By BP again, $|\neg\psi| \cup |\chi| = |\neg\psi \vee \chi|$. Substituting again, we get $|\varphi| = |\neg\psi \vee \chi|$. By BP, we then get $\vdash \varphi \leftrightarrow (\neg\psi \vee \chi)$. So by $\mathrm{RE}_{\mathbf{BA}'}$, we have $\vdash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}(\neg\psi \vee \chi)$. Then by BP again and 2′, we get $\mathbf{B}\mathrm{A}(\neg\psi \vee \chi) \in w$ and $\mathbf{B}\mathrm{A}\psi \in w$. From the former, $\mathbf{B}\mathrm{A}(\psi \rightarrow \chi) \in w$ follows by $\mathrm{RE}_{\mathbf{BA}}$ and BP. But since we are assuming a logic with $\mathrm{K}_{\mathbf{BA}}$, by BP, $\mathbf{B}\mathrm{A}(\psi \rightarrow \chi) \rightarrow (\mathbf{B}\mathrm{A}\psi \rightarrow \mathbf{B}\mathrm{A}\chi) \in w$. And we have $\mathbf{B}\mathrm{A}(\psi \rightarrow \chi) \in w$, and $\mathbf{B}\mathrm{A}\psi \in w$, so by BP, $\mathbf{B}\mathrm{A}\chi \in w$. Finally, by lemma 2$a$, it follows that $|\chi| \in f^1(w)$, that is $Y \in f^1(w)$.

T6^P. Any canonical model for an MDA logic with $\mathrm{FS}_{\mathbf{BA}}$ satisfies constraint ***fs*:** If $X \in P$ & w $\in X$ & $Y \in f^1(w)$, then $X \cap Y \in f^1(w)$. Suppose (1) $X \in P$ & (2) $w \in X$ & (3) $Y \in f^1(w)$. From 1, for some φ, $X = |\varphi|$, and fixing φ, we get from 2, $w \in |\varphi|$, and $\varphi \in w$, by definition of $|\varphi|$ for a canonical model. Given 3, for some ψ, $|\psi| = Y$ & $\mathbf{B}\mathrm{A}\psi \in w$. So we have $\varphi \in w$ & $\mathbf{B}\mathrm{A}\psi \in w$, and hence by BP, $\varphi \wedge \mathbf{B}\mathrm{A}\psi \in w$. But then since $\mathrm{FS}_{\mathbf{BA}} \in w$ & w is an MCS, we have $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \in w$. Then by lemma 2$a$, it follows that $|\varphi \wedge \psi| \in f^1(w)$, that is $|\varphi| \cap |\psi| \in f^1(w)$, that is, $X \cap Y \in f^1(w)$.

T7^P. Any canonical model for an MDA logic with $\mathrm{M}_{\mathbf{BA}}$ satisfies constraint ***m'*:** If $X, Y \in P$ & $X \cap Y \in f^1(w)$, then $X \in f^1(w)$. Suppose first that $X \cap Y \in f^1(w)$. So there exists a χ such that $|\chi| = X \cap$ Y and $\mathbf{B}\mathrm{A}\chi \in w$. Fix χ. By assumption, X, $Y \in P$, and so there is a φ and a ψ such that $|\varphi| = X$ and $|\psi| = Y$. So fixing φ and ψ, $|\varphi| \cap |\psi| = |\chi| = X \cap$ Y; but by BP, $|\varphi| \cap |\psi| = |\varphi \wedge \psi|$, so we have $|\varphi \wedge \psi| = |\chi|$, and then by BP again, we have $\vdash \varphi \wedge \psi \leftrightarrow \chi$. Given $\mathrm{RE}_{\mathbf{BA}}$, it follows that $\vdash \mathbf{B}\mathrm{A}(\varphi \wedge \psi) \leftrightarrow \mathbf{B}\mathrm{A}\chi$, and hence $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \in w$. But given $\mathrm{M}_{\mathbf{BA}}$, by BP, $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \in w$, but w is an MCS, so $(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \in w$. By BP again, $\mathbf{B}\mathrm{A}\varphi \in w$. Then by Lemma 2a), we have $|\varphi| \in f^1(w)$, that is, $X \in f^1(w)$.

T8^P. Any canonical model for an MDA logic with $\mathrm{R}_{\mathbf{BA}}$ satisfies constraint ***r*:** If $X, Y \in P$, $X \cap Y \in f^1(w)$ iff $X \in f^1(w)$ & $Y \in f^1(w)$. This is a corollary of T3 and T7.

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[39] Without the relativization to P in the frames, although we have $|\varphi| = -X \cup Y$ and $\psi = X$, we have no guarantee there is any formula, ψ' such that $|\psi'| = Y$, so that in turn we can get $|\varphi| = -X \cup Y = |\neg\psi| \cup |\psi'| = |\psi \vee \psi'|$. With X, $Y \in P$, this is assured and so strong completeness results. See the Appendix of Goble, Lou 2004 for an encounter with a similar problem, and an alternative strategy that generates weak completeness using the standard semantic constraint for K. The use of P here allows for strong completeness proofs, but only with the constraint involving the relativization to P.

We next turn to correspondences involving candidate axioms for pure dyadic systems.

T9. Any canonical model for any *MDA* logic with T$_{\mathbf{B}A'}$ satisfies constraint ***t′***: If $Y \in f^2(w, X)$, then $w \in Y \cap X$. Suppose $Y \in f^2(w, X)$. Then $\exists\psi(Y = |\psi|\ \&\ \exists\varphi(|\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\psi \in w))$. Fixing ψ, φ, we have: $Y = |\psi|\ \&\ |\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\psi \in w$. But T$_{\mathbf{B}A'}$ is a thesis, so $\mathbf{B}A'_{\varphi}\psi \to (\varphi \wedge \psi) \in w$, and so $(\varphi \wedge \psi) \in w$. So $w \in |\varphi \wedge \psi|$, that is, $w \in |\varphi| \cap |\psi|$, and then the consequent of ***t′*** follows: $w \in Y \cap X$.

T10. Any canonical model for any *MDA* logic with NO′$_{\mathbf{B}A'}$ satisfies the constraint ***no′***: $\neg\exists X$ such that $W \in f^2(w, X)$ and $f^2(w, W) = \varnothing$. For suppose instead that (1) $\exists X[W \in f^2(w, X)]$ or (2) $f^2(w, W) \neq \varnothing$. Fixing X in case 1, we have $W = f^2(w, X)$. But since $|\top| = W$, we have $|\top| \in f^2(w, X)$. So by definition of the canonical models (and since $|\top| = |\top|$), we have $\exists\varphi(|\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\top \in w)$. Fixing φ, we have $|\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\top \in w$. But by schema NO′$_{\mathbf{B}A'}$, we have $\neg(\mathbf{B}A'_{\varphi}\top \vee \mathbf{B}A'_{\top}\varphi) \in w$, and thus $\neg\mathbf{B}A'_{\varphi}\top \in w$, so w is not consistent. In case 2, $X \in f^2(w, W)$, for some X. But since $|\top| = W$, $X \in f^2(w, |\top|)$. So by definition of the canonical models, $\exists\varphi(|\varphi| = X\ \&\ \mathbf{B}A'_{\top}\varphi \in w)$. Fixing φ, we have $|\varphi| = X\ \&\ \mathbf{B}A'_{\top}\varphi \in w$. But given NO′$_{\mathbf{B}A'}$, $\neg(\mathbf{B}A'_{\varphi}\top \vee \mathbf{B}A'_{\top}\varphi) \in w$, and thus $\neg\mathbf{B}A'_{\top}\varphi \in w$, so w is not consistent.

T11. Any canonical model for an MDA logic with AS$_{\mathbf{B}A'}$ satisfies constraint ***as***: If $Y \in f^2(w, X)$, then $X \notin f^2(w, Y)$. Suppose $Y \in f^2(w, X)$. So there exists a ψ and φ such that: $|\psi| = Y$, $|\varphi| = X$ and $\mathbf{B}A'_{\varphi}\psi \in w$. But given AS$_{\mathbf{B}A'}$ and BP, $(\mathbf{B}A'_{\varphi}\psi \to \neg\mathbf{B}A'_{\psi}\varphi) \in w$, and since w is an MCS, $\neg\mathbf{B}A'_{\psi}\varphi \in w$, and so $\mathbf{B}A'_{\psi}\varphi \notin w$. Hence by lemma 2*b*, $|\varphi| \notin f^2(w, |\psi|)$, that is, $X \notin f^2(w, Y)$.

T12. Any canonical model for any MDA logic with IR$_{\mathbf{B}A'}$ satisfies the constraint ***ir***: $X \notin f^2(w, X)$. Suppose $X \in f^2(w, X)$, for some X. Then $\exists\psi(|\psi| = X\ \&\ \exists\varphi(|\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\psi \in w))$. Fix ψ and φ. So $|\psi| = X = |\varphi|\ \&\ \mathbf{B}A'_{\varphi}\psi \in w$. But since $|\psi| = |\varphi|$, $\vdash \psi \leftrightarrow \varphi$, and so by RE$^{r}_{\mathbf{B}A'}$, we have $\mathbf{B}A'_{\varphi}\varphi \in w$. But given IR$_{\mathbf{B}A'}$, $\neg\mathbf{B}A'_{\varphi}\varphi \in w$ too, so w is not consistent.

T13. Any canonical model for any MDA logic with CC$_{\mathbf{B}A'}$ satisfies constraint ***cc***: If $Y \in f^2(w, X)$ and $Z \in f^2(w, X)$, then $Y \cap Z \in f^2(w, X)$. Suppose $Y \in f^2(w, X)$ and $Z \in f^2(w, X)$. Then $\exists\psi(Y = |\psi|\ \&\ \exists\varphi(|\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\psi \in w))$ and $\exists\psi'(Z = |\psi'|\ \&\ \exists\varphi'(|\varphi'| = X\ \&\ \mathbf{B}A'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $Y = |\psi|\ \&\ |\varphi| = X\ \&\ \mathbf{B}A'_{\varphi}\psi \in w$ and $Z = |\psi'|\ \&\ |\varphi'| = X\ \&\ \mathbf{B}A'_{\varphi'}\psi' \in w$. Since $|\varphi| = X\ \&\ |\varphi'| = X$, $|\varphi| = |\varphi'|$, and thus $\vdash \varphi \leftrightarrow \varphi'$. Then given RE$^{l}_{\mathbf{B}A'}$, $\vdash \mathbf{B}A'_{\varphi'}\psi' \leftrightarrow \mathbf{B}A'_{\varphi}\psi'$ and thus $\mathbf{B}A'_{\varphi'}\psi' \leftrightarrow \mathbf{B}A'_{\varphi}\psi' \in w$. So we have $X = |\varphi|\ \&\ Y = |\psi|\ \&\ Z = |\psi'|\ \&\ \mathbf{B}A'_{\varphi}\psi \in w\ \&\ \mathbf{B}A'_{\varphi}\psi' \in w$. But given CC$_{\mathbf{B}A'}$, $(\mathbf{B}A'_{\varphi}\psi \wedge \mathbf{B}A'_{\varphi}\psi') \to \mathbf{B}A'_{\varphi}(\psi \wedge \psi') \in w$, so $\mathbf{B}A'_{\varphi}(\psi \wedge \psi') \in w$. By lemma 2*b*, $\mathbf{B}A'_{\varphi}(\psi \wedge \psi') \in w$ iff $|\psi \wedge \psi'| \in f^2(w, |\varphi|)$. But $|\psi \wedge \psi'| = |\psi| \cap |\psi'|$, that is, $|\psi \wedge \psi'| = Y \cap Z$, and we already have $|\varphi| = X$. So the consequent of ***cc*** follows: $Y \cap Z \in f^2(w, X)$.

The following theorem generalizes the preceding one.

T14. Any canonical model for any MDA logic with DC$_{\mathbf{B}A'}$ satisfies constraint ***dc***: If Y

$\in f^2(w, X)$ and $Y' \in f^2(w, X')$, then $Y \cap Y' \in f^2(w, X \cap X')$. Assume $Y \in f^2(w, X)$ and $Y' \in f^2(w, X')$. Then $\exists\psi(Y = |\psi|$ & $\exists\varphi(|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w))$ and $\exists\psi'(Y' = |\psi'|$ & $\exists\varphi'(|\varphi'| = X'$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $Y = |\psi|$ & $|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$ and $Y' = |\psi'|$ & $|\varphi'| = X'$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. So $Y \cap Y' = |\psi| \cap |\psi'|$, and so by BP, $Y \cap Y' = |\psi \wedge \psi'|$. Similarly, $X \cap X' = |\varphi| \cap |\varphi'| = |\varphi \wedge \varphi'|$. But since $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$ and $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$ and all instances of $\mathrm{DC}_{\mathbf{B}\mathrm{A}'}$ are in w, it follows that $\mathbf{B}\mathrm{A}'_{(\varphi \wedge \varphi')}(\psi \wedge \psi') \in w$. So by lemma 2*b*, $|\psi \wedge \psi'| \in f^2(w, |\varphi \wedge \varphi'|)$, and then from the identities above, $Y \cap Y' \in f^2(w, X \cap X')$ follows.

T15^P. Any canonical model for an MDA logic with $\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***dcs***: <u>If $X, Y \in P$</u> and $X \cap Y \in f^2(w, Z)$ then $X \in f^2(w, Z)$ or $Y \in f^2(w, Z)$. Suppose for *reductio* that (1) $X, Y \in P$, (2) $X \cap Y \in f^2(w, Z)$, and (3) $X \notin f^2(w, Z)$ and (4) $Y \notin f^2(w, Z)$. From 2 we have $\exists\psi(|\psi| = X \cap Y$ & $\exists\varphi(|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w))$, and then fixing ψ, φ, we get (2′) $|\psi| = X \cap Y$ & $|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$. From 3 and 4 it follows that $\neg\exists\psi(|\psi| = X$ & $\exists\varphi(|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w))$ and $\neg\exists\psi(|\psi| = Y$ & $\exists\varphi(|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w))$, that is, (3′) $\forall\psi$(if $|\psi| = X$ then $\forall\varphi(|\varphi| = Z$ only if $\mathbf{B}\mathrm{A}'_\varphi\psi \notin w))$ and (4′) $\forall\psi$(if $|\psi| = Y$ then $\forall\varphi(|\varphi| = Z$ only if $\mathbf{B}\mathrm{A}'_\varphi\psi \notin w))$. From assumption 1 we have $\exists\psi'(|\psi'| = X)$ and $\exists\psi''(|\psi''| = Y)$; instantiating, we have $|\psi'| = X$ and $|\psi''| = Y$. Then applying these to 3′ and 4′, we get (3″)$\forall\varphi$(if $|\varphi| = Z$ then $\mathbf{B}\mathrm{A}'_\varphi\psi' \notin w$) and (4″) $\forall\varphi$(if $|\varphi| = Z$ then $\mathbf{B}\mathrm{A}'_\varphi\psi'' \notin w$). So from $|\varphi| = Z$ in 2′, we have $\mathbf{B}\mathrm{A}'_\varphi\psi' \notin w$ & $\mathbf{B}\mathrm{A}'_\varphi\psi'' \notin w$. Given 2 and the identifications above for X, Y, Z, we have $|\psi'| \cap |\psi''| \in f^2(w, |\varphi|)$, and by BP, it then follows that $|\psi' \wedge \psi''| \in f^2(w, |\varphi|)$. Invoking lemma 2*b*, it follows that $\mathbf{B}\mathrm{A}'_\varphi(\psi' \wedge \psi'') \in w$. But since $|\psi'| = X$ and $|\psi''| = Y$, and $|\psi| = X \cap Y$, it follows that $|\psi| = |\psi'| \cap |\psi''|$, and then from BP, we get $|\psi| = |\psi' \wedge \psi''|$. By BP again, $\vdash \psi \leftrightarrow (\psi' \wedge \psi'')$ follows, and then by $\mathrm{RE}^{\mathrm{r}}_{\mathbf{B}\mathrm{A}'}$, $\vdash \mathbf{B}\mathrm{A}'_\varphi\psi \leftrightarrow \mathbf{B}\mathrm{A}'_\varphi(\psi' \wedge \psi'')$ follows, and then from 2′ and BP, we get $\mathbf{B}\mathrm{A}'_\varphi(\psi' \wedge \psi'') \in w$. So now we have $\mathbf{B}\mathrm{A}'_\varphi(\psi' \wedge \psi'')$ & $\mathbf{B}\mathrm{A}'_\varphi\psi' \notin w$ & $\mathbf{B}\mathrm{A}'_\varphi\psi'' \notin w$, but by $\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$, w is rendered inconsistent.

T16^P. Any canonical model for an MDA logic with $\mathrm{CK}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***ck′***: <u>if $Y \in P$</u>, $-X \cup Y \in f^2(w, Z)$, and $X \in f^2(w, Z)$, then $Y \in f^2(w, Z)$. Assume (1) $Y \in P$ and (2) $-X \cup Y \in f^2(w, Z)$ & $X \in f^2(w, Z)$. So from 2 by clause *d* of *M*, $\exists\psi(|\psi| = -X \cup Y$ & $\exists\varphi(|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w))$ and $\exists\psi'(|\psi'| = X$ & $\exists\varphi'(|\varphi'| = Z$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing φ, ψ, φ', and ψ', we get (3) $|\psi| = -X \cup Y$ & $|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$ and (4) $|\psi'| = X$ & $|\varphi'| = Z$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. From 1 by definition of P, $\exists\chi(|\chi| = Y)$. Fixing χ, we have $|\chi| = Y$, and from 4 by BP, $|\neg\psi'| = -X$. Substituting in 3 we get (3′) $|\psi| = |\neg\psi'| \cup |\chi|$ & $|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$. By BP again, $|\neg\psi'| \cup |\chi| = |\neg\psi' \vee \chi|$. Substituting again in 3′, we get (3″) $|\psi| = |\neg\psi' \vee \chi|$ & $|\varphi| = Z$ & $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$. By BP from 3″, we get $\vdash \psi \leftrightarrow (\neg\psi' \vee \chi)$. So by $\mathrm{RE}^{\mathrm{r}}_{\mathbf{B}\mathrm{A}'}$, we have $\vdash \mathbf{B}\mathrm{A}'_\varphi\psi \leftrightarrow \mathbf{B}\mathrm{A}'_\varphi(\neg\psi' \vee \chi)$. Then from BP and $\mathbf{B}\mathrm{A}'_\varphi\psi \in w$ in 3, we get $\mathbf{B}\mathrm{A}'_\varphi(\neg\psi' \vee \chi) \in w$. But from (3) and (4) we also have $|\varphi| = Z$ and $|\varphi'| = Z$. So by BP, $\vdash \varphi \leftrightarrow \varphi'$, and hence by $\mathrm{RE}^{\mathrm{l}}_{\mathbf{B}\mathrm{A}'}$, we have $\vdash \mathbf{B}\mathrm{A}'_{\varphi'}\psi' \leftrightarrow \mathbf{B}\mathrm{A}'_\varphi\psi'$. And then from BP and $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$ in 4, we get $\mathbf{B}\mathrm{A}'_\varphi\psi'. \in w$. So we have both $\mathbf{B}\mathrm{A}'_\varphi(\neg\psi' \vee \chi) \in w$ and $\mathbf{B}\mathrm{A}'_\varphi\psi' \in w$. Given $\vdash \mathrm{CK}_{\mathbf{B}\mathrm{A}'}$, $\mathrm{CK}_{\mathbf{B}\mathrm{A}'} \in w$, so from BP, it follows that $\mathbf{B}\mathrm{A}'_\varphi\chi \in w$. Then from lemma 2*b*, it follows that $|\chi| \in f^2(w, |\varphi|)$, that is, $Y \in f^2(w, Z)$.

T17. Any canonical model for any MDA logic with $\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***da***: If X

$\in f^2(w, Y)$ and $X \in f^2(w, Z)$, then $X \in f^2(w, Y \cup Z)$. Suppose $X \in f^2(w, Y)$ and $X \in f^2(w, Z)$. Then $\exists\psi(X = |\psi|$ & $\exists\varphi(|\varphi| = Y$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))$ and $\exists\psi'(X = |\psi'|$ & $\exists\varphi'(|\varphi'| = Z$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $X = |\psi|$ & $|\varphi| = Y$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and $X = |\psi'|$ & $|\varphi'| = Z$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. Since $|\psi| = X$ & $|\psi'| = X$, $|\psi| = |\psi'|$, and thus $\vdash \psi \leftrightarrow \psi'$. Then given $\mathrm{RE}^{\mathrm{r}}{}_{\mathbf{B}\mathrm{A}'}$, $\vdash \mathbf{B}\mathrm{A}'_{\varphi'}\psi \leftrightarrow \mathbf{B}\mathrm{A}'_{\varphi'}\psi'$ and thus $\mathbf{B}\mathrm{A}'_{\varphi'}\psi \leftrightarrow \mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. So we have $X = |\psi|$ & $Y = |\varphi|$ & $Z = |\varphi'|$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi \in w$. But given $\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$, $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi) \rightarrow \mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}\psi \in w$, so $\mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}\psi \in w$. By lemma 2*b*, $\mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}\psi \in w$ iff $|\psi| \in f^2(w, |\varphi \vee \varphi'|)$. But $|\varphi \vee \varphi'| = |\varphi| \cup |\varphi'|$, that is, $|\varphi \vee \varphi'| = Y \cup Z$, and we already have $|\psi| = X$. So the consequent of ***da*** follows: $X \in f^2(w, Y \cup Z)$.

The following theorem also generalizes its preceding one.

T18. Any canonical model for any MDA logic with schema $\mathrm{DD}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***dd***: If $Y \in f^2(w, X)$ and $Y' \in f^2(w, X')$, then $Y \cup Y' \in f^2(w, X \cup X')$. Assume $Y \in f^2(w, X)$ and
$Y' \in f^2(w, X')$. Then $\exists\psi(Y = |\psi|$ & $\exists\varphi(|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))$ and $\exists\psi'(Y' = |\psi'|$ & $\exists\varphi'(|\varphi'| = X'$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $Y = |\psi|$ & $|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and $Y' = |\psi'|$ & $|\varphi'| = X'$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. So $Y \cup Y' = |\psi| \cup |\psi'|$, and so by BP, $Y \cup Y' = |\psi \vee \psi'|$. Similarly, $X \cup X' = |\varphi| \cup |\varphi'| = |\varphi \vee \varphi'|$. But since $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$ and using $\mathrm{DD}_{\mathbf{B}\mathrm{A}'}$, it follows that $\mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}(\psi \vee \psi') \in w$. So by lemma 2*b*, $|\psi \vee \psi'| \in f^2(w, |\varphi \vee \varphi'|)$, and then from the identities above, $Y \cup Y' \in f^2(w, X \cup X')$ follows.

T19. Any canonical model for any MDA logic with $\mathrm{TR}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***tr***: If $Y \in f^2(w, X)$ and $Z \in f^2(w, Y)$, then $Z \in f^2(w, X)$. Suppose $Y \in f^2(w, X)$ and $Z \in f^2(w, Y)$. Then $\exists\psi(Y = |\psi|$ & $\exists\varphi(|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))$ and $\exists\psi'(Z = |\psi'|$ & $\exists\varphi'(|\varphi'| = Y$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $Y = |\psi|$ & $|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and $Z = |\psi'|$ & $|\varphi'| = Y$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. Since $Y = |\psi|$ & $|\varphi'| = Y$, $|\psi| = |\varphi'|$, and thus $\vdash \psi \leftrightarrow \varphi'$. Then given $\mathrm{RE}^{\mathrm{l}}{}_{\mathbf{B}\mathrm{A}'}$, $\vdash \mathbf{B}\mathrm{A}'_{\varphi'}\psi' \leftrightarrow \mathbf{B}\mathrm{A}'_{\psi}\psi'$ and thus $\mathbf{B}\mathrm{A}'_{\psi'}\psi' \in w$. So we have $X = |\varphi|$ & $Y = |\varphi'|$ & $Z = |\psi'|$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ & $\mathbf{B}\mathrm{A}'_{\psi}\psi' \in w$. But given $\mathrm{TR}_{\mathbf{B}\mathrm{A}'}$, $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi') \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$, so $\mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$. By lemma 2*b* $\mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$ iff $|\psi'| \in f^2(w, |\varphi|)$. But $|\psi'| = Z$ and $X = |\varphi|$, so the consequent of constraint ***tr*** follows: $Z \in f^2(w, X)$.

T20. Any canonical model for any MDA logic with $\mathrm{CT}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***ct***: If $Y \in f^2(w, X)$ and $Z \in f^2(w, X \cap Y)$, then $Z \in f^2(w, X)$. Suppose $Y \in f^2(w, X)$ and $Z \in f^2(w, X \cap Y)$. Then $\exists\psi(Y = |\psi|$ & $\exists\varphi(|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))$ and $\exists\psi'(Z = |\psi'|$ & $\exists\varphi'(|\varphi'| = X \cap Y$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w))$. Fixing ψ, φ, ψ', and φ', we have: $Y = |\psi|$ & $|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and $Z = |\psi'|$ & $|\varphi'| = X \cap Y$ & $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \in w$. Since $Y = |\psi|$ & $|\varphi| = X$ & $|\varphi'| = X \cap Y$, we have $|\varphi'| = |\varphi| \cap |\psi|$, and thus $|\varphi'| = |\varphi \wedge \psi|$. So $\vdash \varphi' \leftrightarrow (\varphi \wedge \psi)$, and then by $\mathrm{RE}^{\mathrm{l}}{}_{\mathbf{B}\mathrm{A}'}$, $\mathbf{B}\mathrm{A}'_{\varphi'}\psi' \leftrightarrow \mathbf{B}\mathrm{A}'_{(\varphi\wedge\psi)}\psi' \in w$. Hence we have $X = |\varphi|$ & $Y = |\psi|$ & $Z = |\varphi \wedge \psi|$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ & $\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\psi' \in w$. But given $\mathrm{CT}_{\mathbf{B}\mathrm{A}'}$, $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\psi') \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$, so $\mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$. By lemma 2*b*, $\mathbf{B}\mathrm{A}'_{\varphi}\psi' \in w$ iff $|\psi'| \in f^2(w, |\varphi|)$. But $|\psi'| = Z$ and $|\varphi| = X$, so the consequent of constraint ***ct*** follows: $Z \in f^2(w, X)$.

The following candidate dyadic formula involves embedding of a dyadic operator within the scope of a dyadic operator. I call this formula dyadic "stage setting."

T21^{P}.[40] Any canonical model for an MDA logic with $\mathrm{SS}_{\mathbf{B}\mathrm{A}'}$ satisfies the constraint ***ss***: if $X \in P$, $Y \in P$, then if $Z \in f^2(w, U)$ and $Z = \{w'\colon Y \in f^2(w', X)\}$, then $Y \in f^2(w, U \cap X)$. Assume (1) $X \in P$ and $Y \in P$, (2) $Z \in f^2(w, U)$, and (3) $Z = \{w'\colon Y \in f^2(w', X)\}$. Given 2 by clause d of M, $\exists\psi(|\psi| = Z \;\&\; \exists\varphi(|\varphi| = U \;\&\; \mathbf{B}\mathrm{A}'_{\varphi}\psi \in w))$. Fixing φ and ψ, we get $|\psi| = Z \;\&\; |\varphi| = U \;\&\; \mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$. From 1, there must be an χ and χ' with $|\chi| = X$ and $|\chi'| = Y$. Then given $|\psi| = Z$, $|\varphi| = U$, $|\chi| = X$ and $|\chi'| = Y$, from 2 and 3, we get: (2′) $|\psi| \in f^2(w, |\varphi|)$ and (3′) $|\psi| = \{w'\colon |\chi'| \in f^2(w', |\chi|)\}$. But then applying lemma 2b, we get these equivalents: (2″) $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ and (3″) $|\psi| = \{w'\colon \mathbf{B}\mathrm{A}'_{\chi}\chi' \in w'\}$. Then since by definition, $|\mathbf{B}\mathrm{A}'_{\chi}\chi'| = \{w'\colon \mathbf{B}\mathrm{A}'_{\chi}\chi' \in w'\}$, $|\psi| = |\mathbf{B}\mathrm{A}'_{\chi}\chi'|$. Hence by BP, $\vdash \psi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\chi'$, and then by $\mathrm{RE}^{\mathrm{r}}_{\mathbf{B}\mathrm{A}'}$, $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\psi \leftrightarrow \mathbf{B}\mathrm{A}'_{\varphi}\mathbf{B}\mathrm{A}'_{\chi}\chi$, and so $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$ iff $\mathbf{B}\mathrm{A}'_{\varphi}\mathbf{B}\mathrm{A}'_{\chi}\chi' \in w$. But since $\mathrm{SS}_{\mathbf{B}\mathrm{A}'}$ is a thesis, we then get $\mathbf{B}\mathrm{A}'_{(\varphi\wedge\chi)}\chi' \in w$. So by lemma 2$b$, $|\chi'| \in f^2(w, |\varphi \wedge \chi|)$. But by BP, $|\varphi \wedge \chi| = |\varphi| \cap |\chi|$, and since $|\varphi| = U$, $|\chi| = X$ and $|\chi'| = Y$, we have $Y \in f^2(w, U \cap X)$.

T22^{P}. Any canonical model for an MDA logic with $\mathrm{DFS}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***fs*****:** if $X \in P$ & w $\in X$ & $Y \in f^2(w, Z)$, then $X \cap Y \in f^2(w, Z)$. Suppose (1) $X \in P$ & (2) $w \in X$ & (3) $Y \in f^2(w, Z)$. From 1, for some φ, $X = |\varphi|$, and then fixing φ, from 2, $w \in |\varphi|$, and so $\varphi \in w$, by definition of $|\varphi|$ for a canonical model. From 3, we get for some ψ and some χ, $|\psi| = Y$ & $|\chi| = Z$ and $\mathbf{B}\mathrm{A}'_{\chi}\psi \in w$. Fixing ψ and χ, we have $\varphi \in w$ & $\mathbf{B}\mathrm{A}'_{\chi}\psi \in w$ and so by BP, $(\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi) \in w$. But then since DFSBA′ $\in w$ & w is an MCS, by BP we have $\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \in w$. Then by lemma 2$a$, it follows that $|\varphi \wedge \psi| \in f^2(w, |\chi|)$, that is, $|\varphi| \cap |\psi| \in f^2(w, |\chi|)$, that is, $X \cap Y \in f^2(w, Z)$.

T23^{P}. Any canonical model for an MDA logic with $\mathrm{CM}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint **c*m*′**: If $X, Y \in P$ and $X \cap Y \in f^2(w, Z)$, then $X \in f^2(w, Z)$. Suppose first that $X \cap Y \in f^2(w, Z)$. So there exists a χ' and χ such that: $|\chi'| = X \cap Y$, $|\chi| = Z$ and $\mathbf{B}\mathrm{A}'_{\chi}\chi' \in w$. By assumption, X and $Y \in P$, and so there is a φ and a ψ such that $|\varphi| = X$ & $|\psi| = Y$. So $|\varphi| \cap |\psi| = |\chi'| = X \cap \mathrm{Y}$; but by BP, $|\varphi| \cap |\psi| = |\varphi \wedge \psi|$, so we have $|\varphi \wedge \psi| = |\chi'|$, and then by BP again, we have $\vdash (\varphi \wedge \psi) \leftrightarrow \chi'$. Given $\mathrm{RE}^{\mathrm{r}}_{\mathbf{B}\mathrm{A}'}$, it follows that $\vdash \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\chi'$, and so $\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \in w$, since w is a MCS. Given $\mathrm{CM}_{\mathbf{B}\mathrm{A}'}$, by BP, $\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi) \in w$, and hence by BP, $\mathbf{B}\mathrm{A}'_{\chi}\varphi \in w$. Then by Lemma 2$a$, we have $|\varphi| \in f^2(w, |\chi|)$, that is, $X \in f^2(w, Z)$.

T24^{P}. Any canonical model for an MDA logic with $\mathrm{CR}_{\mathbf{B}\mathrm{A}'}$ satisfies constraint ***cr***: If $X, Y \in P$, $X \cap Y \in f^2(w, Z)$ iff $X \in f^2(w, Z)$ & $Y \in f^2(w, Z)$. This is a corollary of T13 and T23^{P}.

The following is the key thesis for combined monadic-dyadic agency logics:

T25. Any canonical model for any *DMA* logic with S satisfies constraint ***s***: If $Y \in f^2(w, X)$, then $X \in f^1(w)$ and $Y \in f^1(w)$. Suppose $Y \in f^2(w, X)$, then $\exists\psi(Y = |\psi| \;\&\; \exists\varphi(|\varphi| = X$

[40] Lou Goble notes in correspondence that T21^{P} can be straightforwardly demonstrated without invoking the relativization to P, *provided that* we are looking at a system that also includes $\mathrm{T}_{\mathbf{B}\mathrm{A}'}$. Given supposition (2), you get $Z = |\psi|$ and $U = |\varphi|$ and $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$. Given $\mathrm{T}_{\mathbf{B}\mathrm{A}'}$, you get $\psi \in w$, hence $w \in |\psi|$. By 3, since $Z = |\psi| = \{w\colon Y \in f^2(w, X)\}$, you get $w \in \{w\colon Y \in f^2(w, X)\}$. So $Y \in f^2(w, X)$. That provides the χ such that $X = |\chi|$ and the χ' such that $Y = |\chi'|$ and the rest continues as before, without invoking assumption 1.

& $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$)). Fixing ψ and φ, we have: $Y = |\psi|$ & $|\varphi| = X$ & $\mathbf{B}\mathrm{A}'_{\varphi}\psi \in w$. But given S, $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \to (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)) \in w$, so $(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \in w$, and thus $\mathbf{B}\mathrm{A}\varphi \in w$ & $\mathbf{B}\mathrm{A}\psi \in w$. By lemma 2*a*, $\mathbf{B}\mathrm{A}\varphi \in$ w iff $|\varphi| \in f^1(w)$ and $\mathbf{B}\mathrm{A}\psi \in w$ iff $|\psi| \in f^1(w)$. So we have the consequent of ***s***: $X \in f^1(w)$ and $Y \in f^1(w)$.

6 Correspondences and completeness for systems summarized

Below is a table summarizing the 25 correspondence theorems:

Formula Schema (and label):	Constraint on Frames
$\mathrm{T}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}\varphi \to \varphi$	***t***) If $X \in f^1(w)$, then $w \in X$
$\mathrm{NO}_{\mathbf{B}\mathrm{A}}$: $\neg\mathbf{B}\mathrm{A}\top$	***no***) $W \notin f^1(w)$
$\mathrm{C}_{\mathbf{B}\mathrm{A}}$: $(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \to \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$	***c***) If $X \in f^1(w)$ and $Y \in f^1(w)$, $X \cap Y \in f^1(w)$
$\mathrm{CS}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \to (\neg\mathbf{B}\mathrm{A}\varphi \to \mathbf{B}\mathrm{A}\psi)$	***cs***) <u>If $X \in P$, $Y \in P$</u>, if $X \cap Y \in f^1(w)$ then $X \in f^1(w)$ or $Y \in f^1(w)$
$\mathrm{K}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \to \psi) \to (\mathbf{B}\mathrm{A}\varphi \to \mathbf{B}\mathrm{A}\psi)$	***k′***) <u>If $Y \in P$</u>, $-X \cup Y \in f^1(w)$ & $X \in f^1(w)$, then $Y \in f^1(w)$
$\mathrm{FS}_{\mathbf{B}\mathrm{A}}$: $(\varphi \wedge \mathbf{B}\mathrm{A}\psi) \to (\mathbf{B}\mathrm{A}(\varphi \wedge \psi)$	***fs***) <u>If $X \in P$</u> & w $\in X$ & $Y \in f^1(w)$, $X \cap Y \in f^1(w)$
$\mathrm{M}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \to (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$	***m′***) <u>If $X, Y \in P$</u> & $X \cap Y \in f^1(w)$, $X \in f^1(w)$
$\mathrm{R}_{\mathbf{B}\mathrm{A}}$: $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$	***r***) <u>If $X, Y \in P$</u>, then $X \cap Y \in f^1(w)$ iff $X \in f^1(w)$ & $Y \in f^1(w)$
$\mathrm{T}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\psi}\varphi \to (\varphi \wedge \psi)$	***t′***) If $Y \in f^2(w, X)$, $w \in Y \cap X$
$\mathrm{NO}'_{\mathbf{B}\mathrm{A}'}$: $\neg(\mathbf{B}\mathrm{A}'_{\varphi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\varphi)$	***no′***) $\neg\exists X(W \in f^2(w, X))$ & $f^2(w, W) = \varnothing$
$\mathrm{AS}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\varphi}\psi \to \neg\mathbf{B}\mathrm{A}'_{\psi}\varphi$	***as***) If $Y \in f^2(w, X)$, $X \notin f^2(w, Y)$
$\mathrm{IR}_{\mathbf{B}\mathrm{A}'}$: $\neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi$	***ir***) $X \notin f^2(w, X)$
$\mathrm{CC}_{\mathbf{B}\mathrm{A}'}$: $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\chi) \to \mathbf{B}\mathrm{A}'_{\varphi}(\psi \wedge \chi)$	***cc***) If $Y \in f^2(w, X)$ and $Z \in f^2(w, X)$, $Y \cap Z \in f^2(w, X)$
$\mathrm{DC}_{\mathbf{B}\mathrm{A}'}$: $(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi') \to \mathbf{B}\mathrm{A}'_{(\varphi \wedge \varphi')}(\psi \wedge \psi')$	***dc***) If $Y \in f^2(w, X)$ & $Y' \in f^2(w, X')$, $Y \cap Y' \in f^2(w, X \cap X')$
$\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \to (\neg\mathbf{B}\mathrm{A}'_{\chi}\varphi \to \mathbf{B}\mathrm{A}'_{\chi}\psi)$	***dcs***) <u>If $X \in P$, $Y \in P$</u> & $X \cap Y \in f^2(w, Z)$, $X \in f^2(w, Z)$ or $Y \in f^1(w, Z)$
$\mathrm{CK}_{\mathbf{B}\mathrm{A}'}$: $\mathbf{B}\mathrm{A}'_{\chi}(\varphi \to \psi) \to (\mathbf{B}\mathrm{A}'_{\chi}\varphi \to \mathbf{B}\mathrm{A}'_{\chi}\psi)$	***ck′***) <u>If $Y \in P$</u>, $-X \cup Y \in f^2(w, Z)$, & $X \in f^2(w, Z)$, then $Y \in f^2(w, Z)$
$\mathrm{DA}_{\mathbf{B}\mathrm{A}'}$: $(\mathbf{B}\mathrm{A}'_{\psi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\varphi) \to \mathbf{B}\mathrm{A}'_{(\psi \vee \chi)}\varphi$	***da***) If $X \in f^2(w, Y)$ and $X \in f^2(w, Z)$, $X \in f^2(w, Y \cup Z)$

DD$_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi') \rightarrow \mathbf{B}\mathrm{A}'_{(\varphi \vee \varphi')}(\psi \vee \psi')$	***dd***)	If $Y \in f^2(w, X)$ and $Y' \in f^2(w, X')$, then $Y \cup Y' \in f^2(w, X \cup X')$
TR$_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\psi}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\chi$	***tr***)	If $Y \in f^2(w, X)$ and $Z \in f^2(w, Y)$, then $Z \in f^2(w, X)$
CT$_{\mathbf{B}\mathrm{A}'}$:	$(\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}\chi$	***ct***)	If $Y \in f^2(w, X)$ & $Z \in f^2(w, X \cap Y)$, then $Z \in f^2(w, X)$
SS$_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}\mathbf{B}\mathrm{A}'_{\varphi}\psi \rightarrow \mathbf{B}\mathrm{A}'_{(\chi \wedge \varphi)}\psi$	***ss***)	<u>If $X \in P$, $Y \in P$</u>, $Z \in f^2(w, U)$ & $Z = \{w' : Y \in f^2(w', X)\}$, then $Y \in f^2(w, \mathrm{U} \cap X)$
DFS$_{\mathbf{B}\mathrm{A}'}$:	$(\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi) \rightarrow \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi)$	***dfs***)	<u>If $X \in P$</u> & $\mathrm{w} \in X$ & $Y \in f^2(w, Z)$, then $X \cap Y \in f^2(w, Z)$
CM$_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi)$	**c*m'***)	<u>If $X, Y \in P$</u> & $X \cap Y \in f^2(w, Z)$, then $X \in f^2(w, Z)$
CR$_{\mathbf{B}\mathrm{A}'}$:	$\mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}'_{\chi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\chi}\psi)$	***cr***):	<u>If $X, Y \in P$</u>, $X \cap Y \in f^2(w, Z)$ iff $X \in f^2(w, Z)$ & $Y \in f^2(w, Z)$
S:	$\mathbf{B}\mathrm{A}'_{\varphi}\psi \rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$	***s***)	If $Y \in f^2(w, X)$, $X \in f^1(w)$ & $Y \in f^1(w)$.

<u>Corollary of FT, Lemma 3, and T1-T25</u>: All of the MDA logics are strongly complete with respect to their intended frames.
<u>Proof</u>: In section 5, for each of the 25 formulae, we have shown that the characteristic semantical constraint associated with that formula must be met by the frame on the canonical model of any logic that contains that formula (independently of what other formulae of the language it contains). Also, in Lemma 3, we showed that any canonical model for an MDA logic is indeed an MDA model. So given the earlier Corollary of FT in section 4 about strong completeness, for any of the MDA logics, strong frame completeness follows for the frames defined by the intended constraints associated with the characteristic schemata for that logic.

Considering only logic specifications by combining schemas (some will be for the same logic, as we've seen)[41], and some combos will be inconsistent, there are 2^7 (128) PMA logic specifications, 2^{15} (32,768) PDA logic specifications, and 2^{23} (8,388,608) DMA logic specifications. Given the fundamental theorem and twenty-five relevant correspondence theorems, strong completeness results follow for all of these that are consistent (do not contain all formulae). We turn these into determination theorems next.

7 Soundness and determination theorems

We now provide the ingredients needed for soundness theorems for any of the specified logics that are consistent; then combining these with the preceding completeness results, we can transform the correspondences to determination theorems for all the

[41] But we ignore R$_{\mathbf{B}\mathrm{A}}$ and R$_{\mathbf{B}\mathrm{A}'}$ since equivalent to M$_{\mathbf{B}\mathrm{A}}$ combined with C$_{\mathbf{B}\mathrm{A}}$, and to M$_{\mathbf{B}\mathrm{A}'}$ combined with C$_{\mathbf{B}\mathrm{A}'}$, respectively.

logics specified that are consistent.[42] In each case where reference to P and Lemma 1 is made, we tag the validity proof with a preceding "^{P}".

We first note that the base PMA logic is just the classical system E. The proofs that modus ponens is validity preserving is familiar and unaffected by the novelty of parameter P in the frames. Similarly for the PDA logics and the DMA (mixed) logics. So we show just that the two replacement rules are validity preserving.

Validity of RE$_{\mathbf{B}\mathrm{A}}$: Suppose $M \vDash \varphi \leftrightarrow \psi$ for every model $M = \langle\langle W, P, f^1\rangle, V\rangle$ on a PMA frame, $\langle W, P, f^1\rangle$. Then $[\![\varphi]\!]^M = [\![\psi]\!]^M$ for every such model and so for any model M and world, w, $[\![\varphi]\!]^M \in f^1(w)$ iff $[\![\psi]\!]^M \in f^1(w)$ and then by [**B**A], $M, w \vDash \mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}\psi$, for every such M.

Validity of RE$^{\mathrm{r}}{}_{\mathbf{B}\mathrm{A}'}$: Suppose $M \vDash \varphi \leftrightarrow \psi$ for every model $M = \langle\langle W, P, f^2\rangle, V\rangle$ on a PDA frame, $\langle W, P, f^2\rangle$. Then $[\![\varphi]\!]^M = [\![\psi]\!]^M$ for every such model and so by [**B**A′], $[\![\varphi]\!]^M \in f^2(w, [\![\chi]\!]^{\mathrm{M}})$ iff $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^{\mathrm{M}}) \in f^1(w)$, for every $w \in W$ of M, and hence $M \vDash \mathbf{B}\mathrm{A}'_{\chi}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi$, for every such M.

Validity of RE$^{\mathrm{r}}{}_{\mathbf{B}\mathrm{A}'}$: (Similarly)

Corollary: All the rules of any of the MDA logics are validity preserving.

We now prove that the constraints that we associated with the 25 schemata in our correspondence theorems validate those schemata.

Constraint ***t*** validates T$_{\mathbf{B}\mathrm{A}}$: Assume $M, w \vDash \mathbf{B}\mathrm{A}\varphi$, for any M, w in any PMA model. So by [**B**A], $[\![\varphi]\!]^M \in f^1(w)$. But then from ***t***, we get $w \in [\![\varphi]\!]^M$, and so $M, w \vDash \varphi$. Hence $M, w \vDash \mathbf{B}\mathrm{A}\varphi \rightarrow \varphi$.

Constraint ***no*** validates NO$_{\mathbf{B}\mathrm{A}}$: Suppose $W \notin f^1(w)$ in every world in every PMA model. Then likewise for $[\top] \notin W$. So by [**B**A], $M, w \nvDash \mathbf{B}\mathrm{A}\top$, and hence $M, w \vDash \neg\mathbf{B}\mathrm{A}\top$ in every world in every PMA model.

Constraint ***c*** validates C$_{\mathbf{B}\mathrm{A}}$: Assume $M, w \vDash \mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi$, so that $[\![\varphi]\!]^M \in f^1(w)$ and $[\![\psi]\!]^M \in f^1(w)$ by [**B**A] and [$\wedge$]. Then by constraint ***c***, $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^1(w)$, that is, $[\![\varphi \wedge \psi]\!]^M \in f^1(w)$, and hence by [**B**A], $M, w \vDash \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$.

^{P}Constraint ***cs*** validates CS$_{\mathbf{B}\mathrm{A}}$: Suppose $M, w \vDash \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$. By [**B**A], $[\![\varphi \wedge \psi]\!]^M \in f^1(w)$. So $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^1(w)$. By Lemma 1, we also get $[\![\varphi]\!]^M \in P$ and $[\![\psi]\!]^M \in P$. Hence from ***cs***, we get $[\![\varphi]\!]^M \in f^1(w)$ or $[\![\psi]\!]^M \in f^1(w)$. So by [**B**A], we get $M, w \vDash \mathbf{B}\mathrm{A}\varphi \vee \mathbf{B}\mathrm{A}\psi$.[43]

[42] This was what turned out to not be possible without modifying the framework in McNamara, Paul 2018. See the next note (on CS$_{\mathbf{B}\mathrm{A}}$ validity) for more specifics.

[43] Here is where the original sort of problem with the stalled completeness proofs reappeared. Without assuring that each formula expresses a proposition in P (see Lemma 1), we would have no assurance that just because $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^1(w)$, that $[\![\varphi]\!]^M$ and $[\![\psi]\!]^M$ were in P, and without that we could not apply ***cs*** to get

^{P}Constraint ***k′*** validates $K_{\mathbf{B}\mathrm{A}}$: Assume $M, w \vDash \mathbf{B}\mathrm{A}(\varphi \to \psi)$ and $M, w \vDash \mathbf{B}\mathrm{A}\varphi$. By [**B**A], ($a$) $[\![\varphi \to \psi]\!]^M \in f^1(w)$, and ($b$) $[\![\varphi]\!]^M \in f^1(w)$. From a, we get $[\![\neg\varphi \vee \psi]\!]^M \in f^1(w)$, and so $-[\![\varphi]\!]^M \cup [\![\psi]\!]^M \in f^1(w)$. By Lemma 1, we also have (c) $[\![\psi]\!]^M \in P$. So we have $[\![\psi]\!]^M \in P$, $-[\![\varphi]\!]^M \cup [\![\psi]\!]^M \in f^1(w)$ and $[\![\varphi]\!]^M \in f^1(w)$. Hence from ***k′***, we get $[\![\psi]\!]^M \in f^1(w)$, and so by [**B**A], $M, w \vDash \mathbf{B}\mathrm{A}\psi$.

^{P}Constraint ***fs*** validates $FS_{\mathbf{B}\mathrm{A}}$: Suppose $M, w \vDash \varphi \wedge \mathbf{B}\mathrm{A}\psi$. So $M, w \vDash \varphi$ and $M, w \vDash \mathbf{B}\mathrm{A}\psi$ From the former, we get (a) $w \in [\![\varphi]\!]^M$ and from the latter by [**B**A], we get (b) $[\![\psi]\!]^M \in f^1(w)$. By Lemma 1, $[\![\varphi]\!]^M \in P$. Hence from ***fs***, we get $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^1(w)$, and so $[\![\varphi \wedge \psi]\!]^M \in f^1(w)$. Then by [**B**A], $w \vDash \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$.

^{P}Constraint ***m′*** validates $M_{\mathbf{B}\mathrm{A}}$: Assume $M, w \vDash \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$. So by [**B**A], we get $[\![\varphi \wedge \psi]\!]^M \in f^1(w)$, and so $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^1(w)$. By Lemma 1, $[\![\varphi]\!]^M \in P$ and $[\![\psi]\!]^M \in P$. Hence from ***m′***, we get $[\![\varphi]\!]^M \in f^1(w)$. So by [**B**A], we get $M, w \vDash \mathbf{B}\mathrm{A}\varphi$. Likewise, by commutation of intersection, we get $M, w \vDash \mathbf{B}\mathrm{A}\psi$, and so. $M, w \vDash \mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi$.

^{P}Constraint ***r′*** validates $R_{\mathbf{B}\mathrm{A}}$: This follows from the preceding proof coupled with the one that ***c*** validates $C_{\mathbf{B}\mathrm{A}}$.

Constraint ***t′*** validates $T_{\mathbf{B}\mathrm{A}'}$: Suppose $M, w \vDash \mathbf{B}\mathrm{A}'_\psi\psi$, for any M, w in any PMA model. So by [**B**A′], $[\![\varphi]\!]^M \in f^2([\![\psi]\!]^M, w)$. Then by ***t′***, $w \in [\![\varphi]\!]^M \cap [\![\psi]\!]^M$ and so $w \in [\![\varphi]\!]^M$ and w $\in [\![\psi]\!]^M$. Hence $M, w \vDash \varphi \wedge \psi$.

Constraint ***no′*** validates $NO_{\mathbf{B}\mathrm{A}'}$: Given ***no′***, for every w and M, and $[\![\varphi]\!]^M$, $W \notin f^2(w, [\![\varphi]\!]^M)$ & $[\![\varphi]\!]^M \notin f^2(w, W)$, that is $[\![\top]\!]^M \notin f^2(w, [\![\varphi]\!]^M)$ & $[\![\varphi]\!]^M \notin f^2(w, [\![\top]\!]^M)$. So by [**B**A′], $M, w \nvDash \mathbf{B}\mathrm{A}'\varphi\top$ & $M, w \nvDash \mathbf{B}\mathrm{A}'_\top\varphi$. Hence $M, w \vDash \neg(\mathbf{B}\mathrm{A}'_\varphi\top \vee \mathbf{B}\mathrm{A}'_\top\varphi)$.

Constraint ***as*** validates $AS_{\mathbf{B}\mathrm{A}'}$: Suppose $M, w \vDash \mathbf{B}\mathrm{A}'_\varphi\psi$. By [**B**A′], we have $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$. From ***as***, we get $[\![\varphi]\!]^M \notin f^2(w, [\![\psi]\!]^M)$, and then from [**B**A′], we easily get $M, w \vDash \neg\mathbf{B}\mathrm{A}'_\psi\varphi$.

Constraint ***ir*** validates $IR_{\mathbf{B}\mathrm{A}'}$: Take any w in any model, M. By ***ir***, for every X, $X \notin f^2(w, X)$; in particular, $[\![\varphi]\!]^M \notin f^2(w, [\![\varphi]\!]^M)$, for any φ. Hence by [**B**A′], $M, w, \nvDash \mathbf{B}\mathrm{A}'_\varphi\varphi$, so $M, w, \vDash \neg\mathbf{B}\mathrm{A}'_\varphi\varphi$.

Constraint ***cc*** validates $CC_{\mathbf{B}\mathrm{A}'}$: Assume $M, w \vDash \mathbf{B}\mathrm{A}'_\varphi\psi \wedge \mathbf{B}\mathrm{A}'_\varphi\chi$. By [**B**A′], $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$ & $[\![\chi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$. Then by constraint ***cc***, $[\![\psi]\!]^M \cap [\![\chi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$, that is, $[\![\psi \wedge \chi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$, and hence by [**B**A′], $M, w \vDash \mathbf{B}\mathrm{A}'_\varphi(\psi \wedge \chi)$.

Constraint ***dc*** validates $DC_{\mathbf{B}\mathrm{A}'}$: Suppose $w \vDash \mathbf{B}\mathrm{A}'_\varphi\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi'$. By [**B**A′], $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$ and $[\![\psi']\!]^M \in f^2(w, [\![\varphi']\!]^M)$. By constraint ***dc***, it follows that $[\![\psi]\!]^M \cap [\![\psi']\!]^M \in f^2(w, [\![\varphi]\!]^M \cap [\![\varphi']\!]^M)$, and so $[\![\psi \wedge \psi']\!]^M \in f^2(w, [\![\varphi \wedge \varphi']\!]^M)$. Hence by [**B**A′], $M, w \vDash \mathbf{B}\mathrm{A}'_{\varphi \wedge \varphi'}(\psi \wedge \psi')$.

$M, w \vDash \mathbf{B}\mathrm{A}\varphi \vee \mathbf{B}\mathrm{A}\psi$. Similarly for validating K (next), and the other validity proofs preceded with a "^{P}".

^{P}Constraint ***dcs*** validates $\text{DCS}_{\mathbf{B}A'}$: Assume $M, w \vDash \mathbf{B}A'_\chi(\varphi \wedge \psi)$. By [$\mathbf{B}$A′], $[\![\varphi \wedge \psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and so it follows that $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. Also, by Lemma 1, $[\![\varphi]\!]^M \in P$ and $[\![\psi]\!]^M \in P$. Hence from ***dcs***, $[\![\varphi]\!]^M \in f^2(w, [\![\chi]\!]^M)$ or $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. So by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_\chi\varphi$ or $M, w \vDash \mathbf{B}A'_\chi\psi$), and so $M, w \vDash (\neg\mathbf{B}A'_\chi\varphi \rightarrow \mathbf{B}A'_\chi\psi)$.

^{P}Constraint ***ck′*** validates $\text{CK}_{\mathbf{B}A'}$: Suppose $M, w \vDash \mathbf{B}A'_\chi(\varphi \rightarrow \psi)$ & $M, w \vDash \mathbf{B}A'_\chi\varphi$. By [$\mathbf{B}$A′], (*a*) $[\![\varphi \rightarrow \psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and (*b*) $[\![\varphi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. From *a*, we get $[\![\neg\varphi \vee \psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and then $-[\![\varphi]\!]^M \cup [\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. Also, by Lemma 1, (*c*) $[\![\psi]\!]^M \in P$. Hence from ***ck′***, we get $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and hence by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_\chi\psi$.

Constraint ***da*** validates $\text{DA}_{\mathbf{B}A'}$: Assume $M, w \vDash \mathbf{B}A'_\psi\varphi \wedge \mathbf{B}A'_\chi\varphi$. By [$\mathbf{B}$A′], $[\![\varphi]\!]^M \in f^2(w, [\![\psi]\!]^M)$ and $[\![\varphi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. So by constraint ***da***, $[\![\varphi]\!]^M \in f^2(w, [\![\psi]\!]^M \cup [\![\chi]\!]^M)$, and so $[\![\varphi]\!]^M \in f^2(w, [\![\psi \vee \chi]\!]^M)$, and hence by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_{(\psi \vee \chi)}\varphi$.

Constraint ***dd*** validates $\text{DD}_{\mathbf{B}A'}$: Suppose $M, w \vDash \mathbf{B}A'_\varphi\psi \wedge \mathbf{B}A'_{\varphi'}\psi'$. By [$\mathbf{B}$A′], $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$ & $[\![\psi']\!]^M \in f^2(w, [\![\varphi']\!]^M)$. So by constraint ***dd***, $[\![\psi]\!]^M \cup [\![\psi']\!]^M \in f^2(w, [\![\varphi]\!]^M \cup [\![\varphi']\!]^M)$, and so $[\![\psi \vee \psi']\!]^M \in f^2(w, [\![\varphi \vee \varphi']\!]^M)$, and hence by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_{(\varphi \vee \varphi')}(\psi \vee \psi')$.

Constraint ***tr*** validates $\text{TR}_{\mathbf{B}A'}$: Assume $M, w \vDash \mathbf{B}A'_\varphi\psi \wedge \mathbf{B}A'_\psi\chi$. By [$\mathbf{B}$A′], $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$ and $[\![\chi]\!]^M \in f^2(w, [\![\psi]\!]^M)$. So by constraint ***tr***, $[\![\chi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$. Hence by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_\varphi\chi$.

Constraint ***ct*** validates $\text{CT}_{\mathbf{B}A'}$: Suppose $M, w \vDash \mathbf{B}A'_\varphi\psi \wedge \mathbf{B}A'_{(\varphi \wedge \psi)}\chi$. By [$\mathbf{B}$A′], $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$ & $[\![\chi]\!]^M \in f^2(w, [\![\varphi \wedge \psi]\!]^M)$, and so by the latter, $[\![\chi]\!]^M \in f^2(w, [\![\varphi]\!]^M \cap [\![\psi]\!]^M)$. So by constraint ***ct***, $[\![\chi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$. Hence by [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_\varphi\chi$.

^{P}Constraint ***ss*** validates $\text{SS}_{\mathbf{B}A'}$: Assume $M, w \vDash \mathbf{B}A'_\chi\mathbf{B}A'_\varphi\psi$. By [$\mathbf{B}$A′], it follows that $[\![\mathbf{B}A'_\varphi\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. By definition, $[\![\mathbf{B}A'_\varphi\psi]\!]^M = \{w' \colon M, w' \vDash \mathbf{B}A'_\varphi\psi\}$, and then by [$\mathbf{B}$A′], it follows that $[\![\mathbf{B}A'_\varphi\psi]\!]^M = \{w' \colon [\![\psi]\!]^M \in f^2(w', [\![\varphi]\!]^M)\}$. Also, by Lemma 1, $[\![\varphi]\!]^M \in P$ and $[\![\psi]\!]^M \in P$. So we have $[\![\varphi]\!]^M \in P$, $[\![\psi]\!]^M \in P$, $[\![\mathbf{B}A'_\varphi\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and $[\![\mathbf{B}A'_\varphi\psi]\!]^M = \{w' \colon [\![\psi]\!]^M \in f^2(w', [\![\varphi]\!]^M)\}$. Hence from ***ss***, we get $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M \cap [\![\varphi]\!]^M)$, and thus $[\![\psi]\!]^M \in f^2(w, [\![\chi \wedge \varphi]\!]^M)$. Finally, by [$\mathbf{B}$A′], we have $M, w \vDash \mathbf{B}A'_{(\chi \wedge \varphi)}\psi$.

^{P}Constraint ***dfs*** validates DFSBA′: Assume $M, w \vDash \varphi \wedge \mathbf{B}A'_\chi\psi$, that is, $M, w \vDash \varphi$ & $M, w \vDash \mathbf{B}A'_\chi\psi$. From the former, we get $w \in [\![\varphi]\!]^M$, and from the latter by [$\mathbf{B}$A′], we get $[\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. Also, from Lemma 1, $[\![\varphi]\!]^M \in P$. So from ***dfs***, $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and so $[\![\varphi \wedge \psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. Then from [$\mathbf{B}$A′], $M, w \vDash \mathbf{B}A'_\chi(\varphi \wedge \psi)$.

^{P}Constraint ***cm′*** validates $\text{CM}_{\mathbf{B}A'}$: Assume $M, w \vDash \mathbf{B}A'_\chi(\varphi \wedge \psi)$. So by [$\mathbf{B}$A′], we get $[\![\varphi \wedge \psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$, and so $[\![\varphi]\!]^M \cap [\![\psi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. Also, by Lemma 1, $[\![\varphi]\!]^M \in P$ and $[\![\psi]\!]^M \in P$. Hence from ***cm′***, $[\![\varphi]\!]^M \in f^2(w, [\![\chi]\!]^M)$. So by [$\mathbf{B}$A′], we get $M, w \vDash \mathbf{B}A'_\chi\varphi$. Likewise, by commutation of intersection, we get $M, w \vDash \mathbf{B}A'_\chi\psi$, and so $M, w \vDash (\mathbf{B}A'_\chi\varphi \wedge \mathbf{B}A'_\chi\psi)$.

^{P}Constraint ***cr*** validates CR$_{\mathbf{B}A'}$: This follows from the preceding proof coupled with the one that ***cc*** validates CC$_{\mathbf{B}A'}$.

Constraint ***s*** validates S: Suppose $M, w \vDash \mathbf{B}A'_{\varphi}\psi$. By [$\mathbf{B}A'$], we have $[\![\psi]\!]^M \in f^2(w, [\![\varphi]\!]^M)$. From ***s***, we get $[\![\varphi]\!]^M \in f^1(w)$ & $[\![\psi]\!]^M \in f^1(w)$, and then from [$\mathbf{B}$A], $M, w \vDash (\mathbf{B}A\varphi \wedge \mathbf{B}A\psi)$.

These theorems demonstrate the validity of the formulae paired with their constraints in our correspondence chart in the prior section. We now remind the reader of the following fact about validity:

> Fact about Validity: If $\varphi_1, ..., \varphi_n$ are formulae respectively valid in any classes of models,
> $C_1, ..., C_n$, then the φ_i are jointly valid in the class of models constituting the intersection of the C_i.[44]

Given that we have shown that the rules of inference in an MDA logic preserve validity, we have the following result:

> *Soundness Theorem:* Each of the MDA logics specified by any consistent combination of the 25 formulae we've considered is sound with respect to the class of MDA models jointly meeting the constraints we've associated with those formulae.

Coupled with our prior correspondence theorems, we get the following determination result:

> Determination Theorem: Each of the MDA logics specified by any consistent combination of the 25 formulae is determined with respect to its associated class of MDA models.

8 Conclusion

With the fundamental theorem for canonical models and the twenty-five correspondence theorems provided above, we have established completeness for a large number of distinct monadic, dyadic, and mixed monadic-dyadic logics, using combinations of the schemata covered in the twenty-five theorems.[45] This is a strength of the framework. The semantic framework is weak in allowing for a great deal of independence between formulae and thus serving well to provide a characterization of

[44] For example, since C$_{\mathbf{B}A}$ is valid in all PMA models satisfying constraint ***c*** and M$_{\mathbf{B}A}$ is valid in all PMA models satisfying constraint ***m***, then C$_{\mathbf{B}A}$ and M$_{\mathbf{B}A}$ are jointly (equivalently, R$_{\mathbf{B}A}$ is) valid in all PMA models satisfying both constraints ***c*** and ***m*** (equivalently, satisfying constraint ***r***).

[45] Although as noted at the end of subsection 2.3, not all combinations will involve non-redundant schemata (schemata not derivable from the remainder), and for all we have shown, some combinations may be inconsistent, in which case soundness and thus determination theorems will not follow, still, the number of consistent such logics is surely vast.

the *domain* of agency logics, at least to a first approximation. Other stronger and more robust semantic frameworks (e.g. like those in the STIT tradition) will in some ways be more philosophically attractive and informative because they provide a more specific and full-bodied model of agency, but at the same time, there is a cost too in greater specificity since it will rule out or interlink formulae in ways that are more substantive and contentious. It is thus useful to have a weaker more general framework as a backdrop or reference point for stronger frameworks, which are of course also worth exploring, but doing so is beyond the scope of the current paper. The result is also a great deal of modularity so that, as has often been done in this agency tradition, one can easily extract any of the agency logics out of the set contained within, as suits one's purposes, and import it into another (e.g. normative) framework where a simple representation of monadic or dyadic agency suffices for the work at hand. As it is, the current framework obviously needs to be expanded to represent monadic and dyadic *ability* and its interactions with monadic and dyadic *agency*. Other natural additional directions would be to add multi-agents, temporal operators, deontic operators, systems with propositional quantifiers, reductive schemes (see Appendix 2 for some indications), as well as exploring other aspects of the metatheory for the framework (e.g. distinctness of logics, decidability), and the alternative strategy of dispensing with P and generating auxiliary canonical models on the fly each time a completeness theorem for the minimal canonical model seems to fail. What we have here is nonetheless a first step toward *systematizing* a modified neighborhood semantic framework for monadic as well as dyadic agency logics, and in a way that facilitates completeness (and determination) theorems unencumbered by the need to devise auxiliary canonical models, seemingly without end.

Appendix 1 on stalled proof

Natural but Failed Attempt at Showing Correspondence of K with Condition (k): We show that in the canonical model M^L for EK, if $W^L{-}X \cup Y \in N^L(w)$ and $X \in N^L(w)$, then $Y \in N^L(w)$. For any w, suppose (a) $X \in N^L(w)$ and (b) $W^L{-}X \cup Y \in N^L(w)$. So by a, $X \in \{|\varphi|: \Box\varphi \in w\}$, and so $X = |\varphi|$, for some φ such that $\Box\varphi \in w$. Fix φ. So $W^L{-}X \cup Y = W^L{-}|\varphi| \cup Y = |\neg\varphi| \cup Y$; then by b, $|\neg\varphi| \cup Y \in \{|\psi|: \Box\psi \in w\}$, that is, $|\neg\varphi| \cup Y = |\psi|$, for some ψ such that $\Box\psi \in w$. [Now what?]
The rub: What can assure that if $|\neg\varphi| \cup Y = |\psi|$ then $\exists\chi$ such that $|\neg\varphi \vee \chi| = |\neg\varphi| \cup Y = |\psi|$? How do we know that Y is expressible by some χ? We need that to use K's presence in L to complete the proof.

A Fix: Suppose instead the frames have an additional parameter, $P \subseteq \text{Pow}(W)$, and then in the canonical model we assure that P is *the* subset of maximal consistent sets of formulae meeting this condition: $X \in P$ iff $\exists\varphi(|\varphi| = X)$, where $|\varphi|$ is the set of maximal consistent sets containing φ.[46]

[46] As we will see, for soundness, we need to tweak the frames so that P has some modest algebraic structure. See the definitions of the frames for MDA logics in section 3. See Cresswell, 1984, 62–64 and note 6 for analogs for Kripke frames.

$M^L = \langle W^L, P^L, N^L, V^L \rangle$ is a canonical model for an E logic iff:
a) W^L = the set of MCSs for L
b) $P^L = \{X: \exists\varphi(|\varphi| = X)\}$, where $X \subseteq W^L$
c) $N^L(w) = \{X: \exists\varphi(|\varphi| = X \,\&\, \Box\varphi \in w)\}$, where $X \subseteq W^L$
d) $V(Pn) = |Pn|$.

The Fundamental Theorem for Canonical Models then goes through straight forwardly. If we then recast the frame condition for formulae K as follows,

(***k′***) <u>If $Y \in P$, then</u> if $W–X \cup Y \in N_{(w)}$ and $X \in N_{(w)}$, then $Y \in N_{(w)}$,

the correspondence proof for EK then goes through smoothly.[47]

Appendix 2 on reductive schemes.

RD: $\mathbf{B}\mathrm{A}\varphi \leftrightarrow \mathbf{B}\mathrm{A}'_{\varphi}\varphi$.

Here is how the monadic formulae we consider above look when recast via RD:

<u>Monadic Formula</u>:		<u>Recast via RD</u>:
$\mathrm{T}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}\varphi \to \varphi$	$\mathbf{B}\mathrm{A}'_{\varphi}\varphi \to \varphi$
$\mathrm{NO}_{\mathbf{B}\mathrm{A}}$:	$\neg\mathbf{B}\mathrm{A}\top$	$\neg\mathbf{B}\mathrm{A}'_{\top}\top$
$\mathrm{C}_{\mathbf{B}\mathrm{A}}$:	$(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \to \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$	$(\mathbf{B}\mathrm{A}'_{\varphi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi) \to \mathbf{B}\mathrm{A}'_{\varphi \wedge \psi}(\varphi \wedge \psi)$
$\mathrm{CS}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \to (\neg\mathbf{B}\mathrm{A}\varphi \to \mathbf{B}\mathrm{A}\psi)$	$\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi) \to (\neg\mathbf{B}\mathrm{A}'_{\varphi}\varphi \to \mathbf{B}\mathrm{A}'_{\psi}\psi)$
$\mathrm{K}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}(\varphi \to \psi) \to (\mathbf{B}\mathrm{A}\varphi \to \mathbf{B}\mathrm{A}\psi)$	$\mathbf{B}\mathrm{A}'_{(\varphi \to \psi)}(\varphi \to \psi) \to (\mathbf{B}\mathrm{A}'_{\varphi}\varphi \to \mathbf{B}\mathrm{A}'_{\psi}\psi)$
$\mathrm{FS}_{\mathbf{B}\mathrm{A}}$:	$(\varphi \wedge \mathbf{B}\mathrm{A}\psi) \to (\mathbf{B}\mathrm{A}(\varphi \wedge \psi)$	$(\varphi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi) \to (\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi)$
$\mathrm{M}_{\mathbf{B}\mathrm{A}}$:	$\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \to (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$	$\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi) \to (\mathbf{B}\mathrm{A}'_{\varphi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi)$
$\mathrm{R}_{\mathbf{B}\mathrm{A}}$**:**	$\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi)$	$\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi) \leftrightarrow (\mathbf{B}\mathrm{A}'_{\varphi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi)$.

Of course, there is potential for deriving monadic formulae via RD and various dyadic formulae. I illustrate a few such derivations here:

$\mathrm{T}_{\mathbf{B}\mathrm{A}'}$ and RD $\Rightarrow$ $\mathrm{T}_{\mathbf{B}\mathrm{A}}$
<u>Proof</u>: Assume $\vdash \mathbf{B}\mathrm{A}'_{\psi}\varphi \to (\varphi \wedge \psi)$. So $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\varphi \to (\varphi \wedge \varphi)$, hence $\vdash \mathbf{B}\mathrm{A}'_{\varphi}\varphi \to \varphi$, and then by RD, $\vdash \mathbf{B}\mathrm{A}\varphi \to \varphi$.

$\mathrm{NO}_{\mathbf{B}\mathrm{A}'}$ and RD $\Rightarrow$ $\mathrm{NO}_{\mathbf{B}\mathrm{A}}$
<u>Proof</u>: Assume $\vdash \neg(\mathbf{B}\mathrm{A}'_{\varphi}\top \vee \mathbf{B}\mathrm{A}'_{\top}\varphi)$. So $\vdash \neg(\mathbf{B}\mathrm{A}'_{\top}\top \vee \mathbf{B}\mathrm{A}'_{\top}\top)$; that is, $\neg\mathbf{B}\mathrm{A}'_{\top}\top$. Hence by RD, $\vdash \neg\mathbf{B}\mathrm{A}\top$.

$\mathrm{DC}_{\mathbf{B}\mathrm{A}'}$ & RD $\Rightarrow$ $\mathrm{C}_{\mathbf{B}\mathrm{A}}$
<u>Proof</u>: Assume $\vdash (\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi'}\psi') \to \mathbf{B}\mathrm{A}'_{(\varphi \wedge \varphi')}(\psi \wedge \psi')$. So $\vdash (\mathbf{B}\mathrm{A}'_{\varphi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\psi}\psi) \to$

[47] See section 5, theorem $\mathrm{T5}^P$.

$\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi)$. Hence by RD, $\vdash (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}\psi) \rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi)$.

There is also potential for mixed cases:

$\mathrm{CC}_{\mathbf{B}\mathrm{A}'}$ & RD $\Rightarrow (\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\psi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}(\varphi \wedge \psi)$
<u>Proof</u>: Assume $\vdash (\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\chi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}(\psi \wedge \chi)$. So in particular, $\vdash (\mathbf{B}\mathrm{A}'_{\varphi}\varphi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\psi)$
$\rightarrow$
$\mathbf{B}\mathrm{A}'_{\varphi}(\varphi \wedge \psi)$, and then by RD, $(\mathbf{B}\mathrm{A}\varphi \wedge \mathbf{B}\mathrm{A}'_{\varphi}\psi) \rightarrow \mathbf{B}\mathrm{A}'_{\varphi}(\varphi \wedge \psi)$.

$\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$ & RD $\Rightarrow \mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\varphi \vee \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\psi)$
<u>Proof</u>: Assume $\vdash \mathbf{B}\mathrm{A}'_{\chi}(\varphi \wedge \psi) \rightarrow (\neg\mathbf{B}\mathrm{A}'_{\chi}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{\chi}\psi)$. So $\vdash \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}(\varphi \wedge \psi) \rightarrow (\neg\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\varphi \rightarrow \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\psi)$. So by RD and sentential logic, $\mathbf{B}\mathrm{A}(\varphi \wedge \psi) \rightarrow (\mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\varphi \vee \mathbf{B}\mathrm{A}'_{(\varphi \wedge \psi)}\psi)$.

The former derived mixed formula is plausible in its own right. The derivability of the latter mixed formula simply reinforces my skepticism about the plausibility of the reduction on the intended interpretation (i.e. it should not follow from $\mathrm{DCS}_{\mathbf{B}\mathrm{A}'}$). However, using RD to explore (especially failed) attempts to prove the monadic formulae we explored via the dyadic analogs we explored does have the benefit of naturally uncovering plausible sounding mixed formulae like that derived in the former proof, which we did not explore in the current paper.

Although beyond the scope of this paper, I believe a reduction of a different sort is more plausible, but it involves propositional quantification:

RD′: $\mathbf{B}\mathrm{A}\varphi \leftrightarrow \exists\psi(\mathbf{B}\mathrm{A}'_{\psi}\varphi \vee \mathbf{B}\mathrm{A}'_{\varphi}\psi)$.

That is, what is brought about by Jane is a consequence of an exercise of her agency or generates a consequence. It is easy to use this reductive scheme along with some independently plausible principles for dyadic agency, and thereby derive various plausible schemes for monadic agency so reduced. It also allows for an easy and plausible characterization of a basic exercise of agency: $\mathbf{B}\mathrm{A}\varphi \wedge \neg\exists\psi\mathbf{B}\mathrm{A}'_{\psi}\varphi$, that is, $\exists\psi\mathbf{B}\mathrm{A}'_{\varphi}\psi \wedge \neg\exists\psi\mathbf{B}\mathrm{A}'_{\psi}\varphi$, per the above reduction (as well as that of the maximal product of an exercise of one's agency, given the addition of a necessity operator as well as propositional quantifiers).

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Paul McNamara
Philosophy Department
University of New Hampshire, USA
paulm@unh.edu