Propositional Variables Occurring Exactly Once in Candidate Modal Axioms

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Abstract

One does not often encounter a proposed axiom for extending one modal logic to another with the following feature: in the axiom in question some propositional variable (sentence letter) appears only once. Indeed, for a large range of modal logics L, which includes all normal modal logics, the sole occurrence of such a sentence letter can be replaced by a propositional truth or falsity constant, to give an arguably simpler axiom yielding the same extension of L, explaining the rarity of such 'variable-isolating' axioms in the literature. But the proof of this simple (and in one form or another, well-known) result – appearing here as Lemma 2.1 – is sensitive to the choice of modal primitives. It breaks down, for example, when, instead of necessity (or possibility), the sole non-Boolean primitive is taken to be noncontingency (or contingency), the main topic of Sections 0 and 4, the latter closing with a selection of the main problems left open. Between these, which we shall have occasion, *inter alia*, to observe that the (routine) proof of the lemma referred to (which is postponed to a final Appendix, Section 5) is also sensitive to the choice of Boolean primitives (Section 3).

0 Pre-Introduction

The present section introduces the theme indicated by the paper's title with some eye-catching illustrations drawn from that approach to modal (propositional) logic in which contingency or noncontingency rather than necessity or possibility is taken as the primitive modal operator. We shall later be concentrating, though with frequent back-reference to the present examples, on the more familiar modal language. That begins with Section 1, which will include a more conventional introduction, mixed in with some preliminary observations, and a more careful explanation of some terminology used casually in the present section. Aspects of the interest of this line of inquiry, for example its potential sensitivity not only to the choice of modal but also to choice of the non-modal (Boolean) primitives, will emerge as we proceed. If the title had not already been lengthy, a subtitle would have been appended – 'scratching the surface', perhaps – to indicate the preliminary nature of the discussion that follows, drawing attention to a phenomenon and making a few observations, while leaving significant questions unanswered (several of which are collected in the final paragraph of Section 4). The surface to be scratched, however, is a wide one, and there should be something of interest for readers with diverse predilections. Those mainly concerned with

Filosofiska Notiser, Årgång 8, Nr 1, 2021, 27–73

noncontingency-based modal logic will find material tailored to that concern especially in the present section, the material after Proposition 3.2 in Section 3, and in Section 4. More conventionally oriented modal logics, with occasional asides on the noncontingency case, occupy most of the remainder, which is written with a non-specialist audience in mind.

In saying that the current section's illustrations are drawn from (non)contingency-based the modal logic, what is meant is this. Instead of treating noncontingency, generally symbolized by means of Δ, as a modal operator defined in terms of a primitive (normal) necessity operator $\Box$ by taking ΔA as $\Box A \vee \Box\neg A$, we take Δ itself as primitive but interpret Δ as though it had been so defined.[1] With $\Box$, should it be present, interpreted by universal quantification over accessible points in a traditional Kripke model, we have ΔA is true at a point if any two accessible points agree on the truth-value of A, and in the absence of (primitive) $\Box$, Δ is taken as primitive with this understanding. Then the associated $\Box$ operator can be recovered by defining $\Box A$ as $\Delta A \wedge A$ if attention is restricted to models with reflexive accessibility relations, though not in general. (This is not to say that $\Box$ is definable in terms of Δ and the Boolean connectives when *and only when* this restriction is in force, as Cresswell [7] observes.)

Looking at recent work by Jie Fan and co-authors (chronologically: [16], [17], [10]), in which several Δ-based (rather than $\Box$-based) logics are discussed, one sees such formulas as the following, using the authors' own labels here, for axiomatizing these logics as extensions of the basic noncontingency logic – basic in the sense that its theorems are the formulas of the Δ-based language true at all points in all models with no conditions on the models' accessibility relations.[2] Double underlining has been used here as an aid to making the point of current interest readily visible: variables occurring exactly once in our sample of candidate axioms have been highlighted in this way:

Δ4: $\Delta p \to \Delta(\Delta p \vee \underline{\underline{q}})$

Δ5: $\neg\Delta p \to \Delta(\neg\Delta p \vee \underline{\underline{q}})$

[1]Similarly, ∇A is a common notation for saying that it is contingent whether A: $\Diamond A \wedge \Diamond\neg A$. Evidently this just amounts to the negation of ΔA (assuming, as we shall, a classical treatment of $\neg$, without which even $\Box$ and $\Diamond$ would not be interdefinable – or more to the point for our current concerns and expressed in terminology introduced in Section 1: would not be linearly interdefinable). Readers may regard it as somewhat distasteful to introduce a special symbol which amounts to the negation of Δ, reminiscent of the plethora of symbols such as N, P, I sometimes used – naming no names – by those whose main interests lie outside of logic, as operators for necessity, possibility and impossibility, by contrast with the use of $\Box$ and $\Diamond$ for necessity and its dual, possibility. This criticism should be softened by the reminder that ∇ is in fact the dual of Δ, $\neg\Delta\neg$, as well as being equivalent to its negation, thanks to the redundancy of the inner occurrence of $\neg$ in the latter prefix. (Compare the case of truth-functional $\leftrightarrow$, where the dual again coincides with the negation. Similarities between material equivalence and noncontingency will come under the spotlight in the second half of Section 3.) Fan [11] has also studied a 'strong noncontingency' operator which applied to A yields something interpreted as $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$; on p. 102 of Humberstone [29], the result of prefixing $\Box$ to the first and second conjuncts are notated respectively as $\Delta^{+}A$ and $\Delta^{-}A$, and regarded along with ΔA as three reasonable pairwise nonequivalent notions of noncontingency in S4, in which logic the conjunction of the first two is equivalent to the third. In what follows, only ΔA is under discussion.

[2]In fact these authors use the corresponding schemata as axiom-schemes rather than specific formulas as here; relevant notational and terminological clarification of this issue will be provided in Section 1.

$\Delta\mathsf{B}$: $p \to \Delta((\Delta p \wedge \Delta(p \to q) \wedge \neg\Delta q) \to \underline{r})$

As the labelling suggests, these function as Δ-based analogues of the familiar modal principles 4, 5 and B, to be put to service in axiomatizing the logics determined by the classes of models whose accessibility relations are respectively transitive, Euclidean and symmetric. That is, calling the basic noncontingency logic K^{Δ} (as in Zolin [59], Fan [10]) and generally, for a normal □-based modal logic L, letting L^{Δ} be the set of formulas in the Δ-based language which belong to L when ΔA is taken as $\Box A \vee \Box\neg A$, we can axiomatize $\mathsf{K4}^{\Delta}$, $\mathsf{K5}^{\Delta}$, KB^{Δ} by extending a suitable axiomatization of K^{Δ} with additional axioms Δ4, Δ5, ΔB, respectively.[3] But here we are not concerned for the moment with those details, so much as with the unusual feature already these formulas exhibit: in each of them some propositional variable appears just once.

It is not every day that one encounters a candidate axiom for (consistently) extending a modal logic, which has this 'isolated variable' feature. Certainly, there are cases in the more familiar setting of □-based normal modal logic, one such case (mentioned in Section 1) being the formula $\Box p$, which can be used to axiomatize the (Post-complete) Verum logic, and some minor variations on this theme, though we shall see in Section 2 some familiar monotonicity-related considerations render such singly occurring sentence letters 'removable' in the sense of Definition 0.1(*i*) below. Now of course for any candidate axiom A we might use to extend one modal logic to another, we could instead use $A \wedge (p_i \vee A)$, or $A \vee (p_i \wedge A)$ to do the same work, with p_i a variable chosen as not occurring in the formula A. One does not in practice encounter such gratuitous complexity, since the second conjunct or disjunct, respectively, in these cases could simply be dropped. The issue of interest, then, is not the *availability* of candidate modal axioms in which some variable has a solitary occurrence, so much as something closer to their *unavoidability*. Even that formulation does not capture the issue precisely, though, since if a propositional variable, q say, does happen to occur exactly once in a formula A, one can typically replace that occurrence by $q \wedge q$ or indeed replace the whole formula A by $A \wedge A$ (or use ∨ instead of ∧ for either of these tricks), again introducing some gratuitous complexity but showing that axioms exhibiting the unusual feature are indeed avoidable and prompting a sharper characterization. We provide this in Definition 0.1(*i*) and in Definition 2.5, introducing something called the isolation property later – though a closer approximation to the 'unavoidability' idea comes later still, with the 'strong isolation property' (Def. 2.9). The former uses the notation "⊕" officially introduced below (Definition 1.3(*iii*)): for the moment think of $\mathsf{L} \oplus A$ as the smallest logic extending L by the inclusion of the formula A, subject to the further condition that provably equivalent formulas in this logic are freely interreplaceable inside other formulas.[4]

[3] "Suitable" here includes a reference not only to the axioms of K^{Δ} but the rules of uniform substitution, Modus Ponens and congruentiality (see Def. 1.3(*i*)). The following minor variant of ΔB, suggested to me by Evgeny Zolin (p.c.) may be easier to work with, and more appealingly reminiscent of the K axiom in □-based modal logic (cf. ΔT in the first paragraph of Section 4): $p \to \Delta(r \to (\Delta(p \to q) \to (\Delta p \to \Delta q)))$.

[4] To apply Definition 1.3(*iii*) in the present setting, reconstrue the notion of a congruential modal logic given in Def. 1.3(*ii*) to be one in which the context $A(p) = \Delta p$ is congruential, the latter understood in accordance with Def. 1.3(*i*).

D 0.1 (*i*) With L and A as in the preceding sentence, the latter having as its propositional variables the sentence letters $q_1, \ldots, q_n$, then q_n is *removable from A* (as a candidate axiom for extending L to $\mathsf{L} \oplus A$) if there is a formula B (of the same language) in which at most the sentence letters $q_1, \ldots, q_{n-1}$ occur and $\mathsf{L} \oplus A = \mathsf{L} \oplus B$. (More generally, q_n is removable from A in the extension of L to $\mathsf{L}^+ \supseteq \mathsf{L}$ when $\mathsf{L}^+ = \mathsf{L} \oplus A$ and $\mathsf{L}^+ = \mathsf{L} \oplus B$ with B as just described.)

(*ii*) When some B as in (*i*) can be found which is a substitution instance of A there, q_n will be said to be removable *by substitution* (from A as an axiom for extending L). ◀

R 0.2 Note that the replacement of q_n by a variable other than $q_1, \ldots, q_{n-1}$ to yield a formula B' does not count as "removing q_n"; since all logics under consideration in our discussion will be assumed to be closed under uniform substitution, we have $\mathsf{L} \oplus A = \mathsf{L} \oplus B$ and ignoring the proviso about using only the remaining variables in A in Def. 0.1(*i*) would trivially render every variable in A, whether it appears exactly once or more than once, removable from A as an axiom for extending L, and indeed removable by substitution. (This would mean that no extension of one logic to another would possess what Definition 2.5 below calls the isolation property.) ◀

Definition 0.1(*i*) is not restricted to the case in which q_n occurs exactly once in A, though its interest for us here resides in that particular case. In a large range of $\Box$-based modal logics, when one such logic, L, is extended to another, $\mathsf{L} \oplus A$, and the formula A features some propositional variable exactly once (however many other variables occur however many times in A) then that variable is removable by substitution. Theorem 2.2 gives the general idea, though we may remove any of the Boolean primitives listed there or add any others which are, as the following section puts it, 'linearly definable' in terms of the primitives. The examples above (Δ4 etc.) suggest that this is considerably less so for the case of Δ-based modal logics, and we can indeed easily see this to be so in the following case.

E 0.3 Consider the formula $\Delta\Delta p$ as a candidate axiom for extending the basic noncontingency logic K^Δ. Since p is the only sentence letter occurring in the formula, to make its removability even provisionally a live option, we need to make sure we have the truth and falsity constants $\top$ and $\bot$ available in the language (as we shall in Section 1), or there will be no formulas available to play the B role in Definitions 0.1 (when A is $\Delta\Delta p$). But it is not hard to see that for any formula B constructed from these nullary connectives by means of the non-nullary Boolean connectives and Δ, B is either already in K^Δ or else $\neg B$ is.[5] So in neither case do we get a consistent proper extension of the basic system, as we do with $\Delta\Delta p$.[6] We return to this formula below after a word on the earlier Δ examples. ◀

[5]Note the contrast with $\Box$-based K, which does not similarly decide all such 'pure' formulas (as it is put in Humberstone [29], p. 171*f*).

[6] The consistency of this extension of K^Δ is an immediate consequence of the following observation – the 5$\flat$ case of Theorem 5.1 in Zolin [59], the relevant part of the proof beginning at the base of p. 544 there – for the statement of which we use this notation: given a frame $\langle W, R\rangle$ for $u \in W$ we write $R(u)$ for the set of $v \in U$ such that Ruv and $|R(u)|$ for the cardinality of that set. Then the result in question (inessentially reformulated) reads: $\Delta\Delta p$ is valid on a frame $\langle W, R\rangle$ if and only if for all $w \in W$, $x, y \in R(w)$, if $|R(x)| \geq 2$, then $R(x) = R(y)$.

In the case of $\Delta 4$ and $\Delta 5$ above, one might suppose that the conspicuously solitary q could simply be dropped, giving $w\Delta 4$ and $w\Delta 5$ here:

$$w\Delta 4: \quad \Delta p \to \Delta\Delta p \qquad\qquad w\Delta 5: \quad \neg\Delta p \to \Delta\neg\Delta p,$$

since, in the case of $w\Delta 4$ at least, this was, after all, the simpler form used in the original venture into (non)contingency-based modal logic, Montgomery and Routley [39].[7] (See also these pioneering authors' [40].) Here the labels continue to be those of Fan and co-authors – "w" suggesting a *weakening* of the principle in question – and all these candidate axioms can be found on p. 86 of [17], to which the reader interested in seeing how, when reflexivity is not being assumed, the "w" forms of $\Delta 4$ and $\Delta 5$ are indeed *too* weak to allow for the derivation of the 'unweakened' version needed for completeness (w.r.t. the classes of transitive and Euclidean frames) when appended to the basic system. (detailed references to pertinent earlier work by Kuhn, Schumm, Zolin on these topics can be found on p. 87 in [17][8]). It begins to look, then, as though q may not be removable from $\Delta 4$ and $\Delta 5$ as axioms extending the basic system – though, by contrast with the unremovability of p from $\Delta\Delta p$, these considerations are merely suggestive rather than conclusive.

The intention (of Fan et al.) behind $w\Delta 4$ and $w\Delta 5$ is presumably to have the forms resemble those of the $\Box$-based axioms 4 and 5, with Δ replacing $\Box$;[9] otherwise it would pointless to write $\Delta\neg\Delta p$ as the consequent of $w\Delta 5$, since we could equivalently and more simply write this consequent as $\Delta\Delta p$. In fact Montgomery and Routley, wanting to axiomatize a Δ-based version of S5 opted for this simplified consequent, $\Delta\Delta p$ (encountered in the Example 0.3 for potentially extending K^{Δ} rather than, as with Montgomery and Routley, extending KT^{Δ}), and didn't bother with the antecedent at all, giving us something that concisely conveys the key idea: for any statement, it is noncontingent whether that statement is noncontingent.[10] Since $\Delta\Delta p$ is the consequent of each of $w\Delta 4$ and (the rewritten form of) $w\Delta 5$, we have the "w" formulas as simple truth-functional consequences of $\Delta\Delta p$. And since the antecedents of $w\Delta 4$ and

[7] Original *published* venture, to be precise, in view of Lemmon and Gjertsen's 1959 axiomatization of $\mathsf{S5}^{\Delta}$, described under 11.24 on p. 312 of Prior [45]. Curiously enough, [39], on its second page, casually mentions "the Lemmon–Gjertsen formulation of S5" without giving the reader any idea where this axiomatization can be found. Prior's rendering of it is included in Section 4 below.

[8] It is worth mentioning here, since it is potentially confusing, that by contrast with Fan, Zolin (in [59], the relevant paper), calls what are essentially $\Delta 4$ and $\Delta 5$ "weak transitivity" and "weak Euclideanness", respectively, and calls $w\Delta 4$ $\flat$-transitivity, using the label "$\flat$-Euclideanness" for the formula $\Delta\Delta p$ of Example 0.3. Zolin's reference [6], apparently to an Abstract by Kuhn, should be to the relevant review by Schumm in *Mathematical Reviews*/*MathSciNet* of Kuhn [32].

[9] The "non-w" axioms cited above from [17], are described there (p. 86) as chosen to "get as close as possible to the 'translation' of the standard modal logic axioms, with the help of AD," where this last is a reference to something the authors call the 'Almost Definability' schema discussed in [16] and [17]. (AD was incorrectly transcribed in the final line of note 217 on p. 254 of Humberstone [29]: the second occurrence of B there should be "A". As is also mentioned in that footnote, there is a mistake in the would-be proof in Humberstone [28] of an incorrect normality claim for the logic of a necessity-like operator defined by Zolin in terms of noncontingency. This mistake was found by Zolin and is diagnosed and explained by him, along with much else, in Zolin [60].)

[10] As Montgomery and Routley [39], p. 327, put it: "The S5 axiom $\Delta\Delta p$ reveals especially clearly the interpretation of the modalities of S5". We could use the equivalence of the prefixes Δ and $\Delta\neg$ to write this as $\Delta\nabla p$: it is noncontingent whether the statement in question is contingent ("∇" here, from note 1).

(rewritten) $w\Delta 5$ are each other's negations, together they have the consequent, $\Delta\Delta p$, as a truth-functional consequence; $\Delta\Delta p$ is thus truth-functionally equivalent to the conjunction of the two conditional axioms. (Loose ends and further considerations on the subject of Δ-based modal logic are taken up under in Section 4, including KT^{Δ} and its extensions.)

1 Isolated Variables

Parking the relatively unfamiliar setting of Δ-based modal logic in the background for now, let us build up to the more familiar □-based setting with some more careful official introductions of the notions in play. We consider propositional languages here, all sharing the same countable supply of propositional variables (or sentence letters), $p_1, p_2, \ldots, p_n, \ldots$ (the first three of which will be abbreviated to p, q, r) and differing at most in respect of their stock of connectives, with $\#(A_1, \ldots, A_m)$ a formula of a language with m-ary connectives # and formulas $A_1, \ldots, A_m$. As usual, when $m = 0, 1, 2$ this is written as #, $\#A_1$ or $A_1 \# A_2$, respectively.[11] As is evident from this paragraph as well as Section 0, capital roman letters, sometimes decorated with numerical subscripts, serve as schematic letters (metalinguistic variables over arbitrary formulas, that is). Below, we make use of consequence relation notation "$\vdash_{\mathsf{L}}$" where the consequence relation in question is defined in terms of (the set of formulas) L.

D 1.1 (*i*) A formula A is *linear* if every propositional variable occurring in A occurs exactly once; a schema is linear if every schematic letter appearing in it appears exactly once.

(*ii*) A is an *isolating* formula if some propositional variable occurring in A occurs exactly once, and any variable having such a solitary occurrence there is said to be *isolated* in A; as in (*i*), we extend the terminology to schemata by replacing the reference to propositional variables by one to schematic letters.

(*iii*) The definition of a connective in terms of primitive connectives is a *linear* definition if the defining schema is a linear schema. (Clarification and examples follow.) ◀

Thus any linear formula other than one constructed by the application of m-ary connectives ($m \geq 1$) to sentential constants (0-ary connectives) is an isolating formula, and, any variables occurring in a linear formula are isolated in that formula. Definitions 1.1 are sensitive to the division into primitive and defined (or 'derived') connectives, and here we take defined connectives to be metalinguistic abbreviations. Thus in the course of a presentation of classical propositional logic, the definition

$$A \to B =_{\mathsf{Df}} \neg A \vee B,$$

[11]In Definition 0.1(*i*), we made an ad hoc use of "$q_1, \ldots, q_n$" for any sequence of n sentence letters, not wanting to write "$p_1, \ldots, p_n$" with its suggestion that these comprise specifically the first n in the official enumeration of such letters.

is a declaration of intent to refer, when convenient, to the formula $\neg q \vee \neg(p \vee \neg r)$, for example, as $q \to \neg(p \vee \neg r)$.[12] Since the *definiens*, as on the right of the example just inset, is a linear schema, Def. 1.1(*iii*) rules that here we have a linear definition, and we shall refer to the derived (or defined) connective as linearly definable in terms of the primitives used in the definiens. (The left-hand side will always be a linear schema because of the standard conditions on defining an *n*-place predicate letter, function symbol, or connective: the *definiendum* needs to be given in the most general setting, rather than in some special case, such as with the first and second arguments identified.) The Russell–Łukasiewicz definition of disjunction in terms of implication

$$A \vee B =_{\mathsf{Df}} (A \to B) \to B$$

is a non-linear definition, revealing the sensitivity of the concepts introduced in Definitions 1.1 to the choice of logical primitives.[13] With $\vee$ primitive, $p \vee q$ is an linear formula in which each of p, q is isolated, while with $\vee$ defined as above, $p \vee q$ is not a linear formula and it isolates only the variable p. Modal examples of non-linear definitions were conspicuous in Section 0: defining ΔA as $\Box A \vee \Box\neg A$ and (for extensions of KT^{Δ}) $\Box A$ as $\Delta A \wedge A$ evidently fall into this category, making Δp a linear formula when Δ is the only non-Boolean primitive and a non-linear formula when $\Box$ (or the linearly interdefinable $\Diamond$) is the only non-Boolean primitive. The first of these cases of non-linear definition will be seen to have special relevance to the removability of isolated variables in favour of constants because it not only wraps up two occurrences of any variable in the $\Box$-based formulation into a single occurrence in the Δ-based formulation, but does so in such a way that of the two occurrences in question, one is positive and the other negative, as these terms are explained in the following section. Finally, apropos of Definition 1.1(*ii*), note that the terms *isolating* and *isolated* also have a technical use in first-order model theory (in connection with the Omitting Types theorem) which has nothing to do with their use here.[14]

Other than as it appears in the explaining what makes a definition linear, the notion of linear formula will not occupy our attention in what follows as much as the notion of an isolating formula, but we pause to register a characteristic feature of linear formulas:

P 1.2 *No linear formula constructed via connectives from at most $\wedge$, $\vee$, $\to$, $\leftrightarrow$, $\neg$ is a classical tautology.*

For an explanation as to why this is so (and of how to extend the list of connectives mentioned indefinitely), see the discussion in §9.2 of [26] leading up to Corollary

[12]For present purposes, this means that we are adopting the metalinguistic view, rather than the object-linguistic view, of what a defined connective is, as these are distinguished in Humberstone [26] §3.6 and p. 443, base.

[13]The same applies in principle to other such concepts defined in terms of number of occurrences of propositional variables/sentence letters, or claims couched in terms of such concepts, though perhaps the best known examples of such claims concern pure implicational logics, concerning which, in practice, such issues have accordingly not been attended to: Jaśkowski [31], Belnap [2]. Additional references can be found under the heading ‘The 2-Property and the 1,2-Property’ on p. 1127 of my [26], where Jaśkowski’s paper (being known to me only through a translation provided by J. R. Hindley) is mentioned though not explicitly cited.

[14]As in, for example, Rothmaler [49] p. 196, top para.

9.27.4. Note that the nullary truth and falsity constants, $\top$ and $\bot$, cannot be added to the list in Proposition 1.2 (consider for example the linear tautology $\bot \to p$).[15] From Proposition 1.2 we can infer that no linear formula in the connectives there listed can be intuitionistically provable, and also that, returning to classical logic for those connectives, if the modal operator $\Box$ is added to the list (or $\Diamond$ instead, or, for that matter, as well) we can conclude that no modal logic which is a sublogic of the 'trivial' modal logic (KT! in the nomenclature of Chellas [6]) – which includes such favourites as S4 and S5 – can have any linear theorems constructed using only connectives on the list thus extended.[16]

Isolating formulas are, by contrast, very familiar among the *theorems* of the logics just mentioned, as formulated with any frequently chosen set of primitive connectives, such as $\Box p \to \Box(q \to p)$, isolating q and provable in any normal (or indeed any monotonic) modal logic – definitions follow presently – and isolating the variable q, or again $\Box p \to (\Box q \to \Box p)$, provable in all modal logics, or again the result of deleting the occurrences of $\Box$ in either of these, provable in intuitionistic (and therefore also classical) propositional logic. However, they are far less frequently encountered, to stick with the modal case, as candidate axioms for properly extending (e.g.) the smallest normal modal logic K (or again, more pertinently, the smallest monotonic modal logic EM).[17]

Let us pause to explain some of the terminology just used, to keep the discussion relatively self-contained. We take logics here to be (certain) sets of formulas, and sometimes speak of membership in a logic so conceived as provability in that logic. It is convenient also to help ourselves to some 'consequence relation' notation for the sake of such explanations, so we write, where Γ is a set of formulas of the language of some logic L, $\Gamma \vdash_{\mathsf{L}} B$ to mean that, for some $A_1, \ldots, A_n \in \Gamma$, the formula

$$A_1 \to (A_2 \to \ldots \to (A_n \to B)\ldots)$$

is an element of – or (as we typically say) is provable in or is a theorem of – the logic L. This presumes that the language of L has a (primitive or defined) binary connective $\to$ with suitable properties, and in all the cases we shall consider those properties in fact suffice for the relation $\vdash$ between sets of formulas and individual formulas to be a finitary consequence relation, and indeed, because of the earlier restriction to L

[15] The cited passage in [26] also gives a reason for preferring the present terminology over that of Pahi [34], in which linear formulas are called 'variable-like' formulas.

[16] On the non-modal but classical front, the reasoning alluded to establishes the following version of Proposition 1.2 in a more austere setting: no linear formula constructed using only the Sheffer stroke ('nand') is a classical tautology. (Likewise for the case of 'nor'.) While this means that any tautology in this language must contain at least two occurrences of some variable, in the present case something stronger is true: any tautology must contain at least *three* occurrences of some variable.

[17] Similarly, demodalizing the example just given, we can say that the formula $p \to (q \vee p)$ and similar cases, such as $p \to (q \to p)$, $(p \wedge q) \to p$ and so on, would not be at all unexpected in an axiomatization of classical propositional logic, or indeed intuitionistic propositional logic. A non-modal version of the present discussion might concern candidate axioms for intermediate logics: for which proper extensions (of intuitionistic propositional logic) by means of axioms $A(q)$, in which q occurs exactly once, is q removable? As in the classical case, there is a sensitivity to the choice of primitives, so for definiteness one might consider this question, in the first instance, relative to a familiar set. Theorem 2.2 below can be proved for the set of connectives listed there (though interpreted as governed by intuitionistic logic) in the same way as there. But as to what happens if $\to$ in that list there is replaced by $\leftrightarrow$, for example – rendering $\to$ (non-linearly) definable – would remain to be discovered. For more on intuitionistic and intermediate logics, see notes 29 and 64.

closed under uniform substitution, a substitution-invariant such consequence relation. Following the usual convention, "$\{A_1, \ldots, A_n\} \vdash_{\mathsf{L}} B$" is written as "$A_1, \ldots, A_n \vdash_{\mathsf{L}} B$" and "$\varnothing \vdash_{\mathsf{L}} B$" as "$\vdash_{\mathsf{L}} B$". We write $A \dashv\vdash_{\mathsf{L}} B$ to mean "$A \vdash_{\mathsf{L}} B$ and $B \vdash_{\mathsf{L}} A$", and regard a formula $A(p_i)$ in which a particular sentence letter p_i occurs (perhaps among others) as a (1-ary) *context* in which B occurs in the result, $A(B)$ or substituting B for all occurrences of p_i in $A(p_i)$.[18] Then we have

D 1.3 (*i*) The context $A(p_i)$ is *congruential*, *monotone*, *antitone*, or *normal* in L according as: $B \dashv\vdash_{\mathsf{L}} C$ implies $A(B) \dashv\vdash_{\mathsf{L}} A(C)$ for all B, C; $B \vdash_{\mathsf{L}} C$ implies $A(B) \vdash_{\mathsf{L}} A(C)$ for all B, C; $B \vdash_{\mathsf{L}} C$ implies $A(C) \vdash_{\mathsf{L}} A(B)$ for all B, C; or $B_1, \ldots, B_n \vdash_{\mathsf{L}} C$ implies $A(B_1), \ldots, A(B_n) \vdash_{\mathsf{L}} A(C)$, for all $B_1, \ldots, B_n, C$ $(n \geq 0)$.

(*ii*) For L with 1-ary connective $\Box$ as its sole non-Boolean primitive, L is a *modal logic* for some functionally complete set of Boolean connectives, if L contains all truth-functional tautologies in those connectives and is closed under Modus Ponens (for $\to$) as well as uniform substitution, and L is a *congruential*, *monotone*, *antitone* or *normal* modal logic according as the context $A(p) = \Box p$ is congruential, monotone, antitone or normal in L.

(*iii*) For any modal logic L and any set of formulas Γ: $\mathsf{L} \oplus \Gamma$ is the smallest congruential modal logic containing all formulas in $\mathsf{L} \cup \Gamma$. If $\Gamma = \{A\}$ we write this as $\mathsf{L} \oplus A$ rather than $\mathsf{L} \oplus \{A\}$. ◀

Each of (*i*)–(*iii*) here calls for comment.

R 1.4 (*i*) In practice, and for the sake of familiarity when speaking of modal logics rather than modal operators or contexts, we will use the adjectives *monotonic* and *antitonic* rather than 'monotone' or 'antitone'. There is, after all, little danger of confusion with the unrelated 'monotonic/nonmonotonic logics' as inference relations satisfying or not satisfying the 'weakening' or 'monotonicity' condition (whose satisfaction is required for them to be consequence relations in the narrow sense in which that phrase is used above). Also monotone and antitone contexts are often called upward and downward monotone, respectively (or again: *entailment-preserving* and *entailment-reversing* contexts). We have omitted from Definitions 1.3 the concept of regularity, intermediate between monotonicity and normality, figuring prominently in Chellas [6] – cf. also note 23 – and obtained by restricting the normality condition to the case of $n = 2$ (or equivalently, to $n \geq 2$). Stronger conditions that those singled out in Definition 1.3(*i*), and thus stronger than those appearing in 1.3(*ii*), are obtained by allowing side-formulas. (Here we take advantage of the classical underpinnings for the logics under current consideration. More generally suitable definitions see [26], p. 491*f*. Thus $A(p)$ is *monotone with side-formulas* in L when $D, B \vdash_{\mathsf{L}} C$ implies

[18] One may find the "$A(p)$" notation here objectionably obscure and ambiguous. We say that $A(p)$ or $A(q)$ etc. is a formula, but we treat is as a context by the very act of highlighting one from among possibly several sentence letters it contains. It is possible to use a much less casual notation, as Timothy Williamson explains on the second page of his [58]. Instead of "$A(p)$" for a context ("of one variable", we should say for specificity), Williamson favours the ordered pair notation $\langle A, p\rangle$ for a context, where A really is just a formula and the sentence letter in the second position indicates what is to be replaced uniformly in A by the formula B to give "the result of putting B in the context $\langle A, p\rangle$".

$D, A(C) \vdash_{\mathsf{L}} A(B)$ for all B, C, D, and similarly with *antitone*.[19] $A(p)$ is *congruential with side-formulas*, more commonly called *extensional* (in sentence position), when for all B, C, D: $D, B \vdash_{\mathsf{L}} C$ and $D, C \vdash_{\mathsf{L}} B$ imply $D, A(C) \vdash_{\mathsf{L}} A(B)$, and therefore also $D, A(B) \vdash_{\mathsf{L}} A(C)$ in view of the symmetrical formulation.)

(*ii*) Two points arise in connection with Definition 1.3(*ii*). The first is that a more explicit terminology would say "monomodal" logic, since there is only one non-Boolean primitive connective. The discussion in Section 0 used the notation Δ for the sole such primitive so this can be regarded officially as just a suggestive way of writing $\Box$ when the intended interpretation is as expressing noncontingency, understood as $\Box' A \vee \Box' \neg A$ for *another*, typically normal, Box operator $\Box'$ – which we promptly expel from the language when pursuing what we called, and will continue to call, Δ-based modal logic. Second point: the notion of functional completeness employed in Definition 1.3(*ii*) should for convenience here be taken as strong rather than weak functional completeness ([29], p. 12 for instance), which in practice means that the nullary $\bot$ or $\top$ should be among the primitive connectives – as mentioned in Example 0.3 – with, for example (in the congruential setting), $A \wedge \neg A$ not being an acceptable *definiens* for $\bot$ having as its associated semantic value the 1-place constant false truth-function rather than the 0-place truth-function – i.e., truth-value – *false*. In fact we will presume that *both* $\top$ and $\bot$ are among the primitives for convenience (e.g., in the formulation of the 'Proto-Question' below), though given the presumed omnipresence of $\to$ as a primitive could just as well take $\top$ as abbreviating $\bot \to \bot$. As this illustrates, we do not restrict attention to functionally complete sets of Boolean connectives which are minimal (irredundant) in that regard. (It should be added that modal logic on a functionally incomplete Boolean basis itself turns up several interesting phenomena – see Humberstone [22], Dunn [8] – though these do not concern us here.)

(*iii*) Readers accustomed to thinking of $\mathsf{L}+\Gamma$ as the smallest modal logic including $\mathsf{L} \cup \Gamma$ and of $\mathsf{L} \oplus \Gamma$ as the smallest normal modal logic including $\mathsf{L} \cup \Gamma$ (as in Humberstone [29] for instance), and "$\mathsf{L} \oplus \{A\}$" abbreviated to "$\mathsf{L} \oplus A$", will not have their expectations disrupted when $\mathsf{L} \supseteq \mathsf{K}$, since K's normal extensions are precisely its congruential extensions. ("K" explained immediately below.) ◀

As anticipated in the second paragraph after Proposition 1.2 above, and following [6], the smallest congruential, monotonic, and normal modal logics are called E, EM and K. And as remarked in that paragraph, one seldom encounters isolating formulas A used to axiomatize (consistent) normal modal logics L in the sense of specifying L as the smallest normal modal logic containing the formula A, for the case of A as an isolating formula (or containing all instances of a corresponding isolating schema, for that matter). A well-known exception, mentioned in Section 0, would be the 'Verum' logic for which $\Box p$ be such an A, and indeed more generally with $\Box^m p$ playing that

[19] In a more general setting one would allow multiple side-formulas on the left and on the right – letting $\Vdash$ be a generalized (or 'multiple-conclusion') consequence relation and say that $A(p)$ is monotone with side-formulas according to $\Vdash$ when for arbitrary (here assumed finite) sets of formulas Γ and Σ, if $\Gamma, B \Vdash C, \Sigma$ then $\Gamma, A(B) \Vdash A(C), \Sigma$, but in the present classical (and functionally complete) setting, we are just using "D" in the definition to do duty for the conjunction whose conjuncts are the formulas in Γ and the negations of the formulas in Σ.

role for any $m \geq 1$.[20] But more commonly, with $\bot$ available as a nullary connective, one could equally well describe these logics as the smallest normal modal logics to contain $\Box\bot$ (or in the general case, $\Box^m\bot$), so as well as the isolating (and indeed also linear) axiom $\Box p$ we have the simpler – if reducing the number of sentence letters without increasing the length of the formula is regarded as simplifying – variable-free alternative axiomatization(s) using $\bot$.[21]

The same applies in the case of Prior's 'ending time' axiom $\Box p \vee \Diamond\Box q$, with its two isolated variables.[22] Again here we have an isolating (and linear) formula which can be simplified to the variable-free $\Box\bot \vee \Diamond\Box\bot$ axiomatizing the same normal modal logic. In this case there is also, as Prior notes, the $\bot$-free non-isolating (and non-linear) alternative $\Box p \vee \Diamond\Box p$ – or with q instead, of course, providing an example of removal by substitution (Def. 0.1(*ii*)) in which it is not a constant that is substituted for the isolated variable (taken that to be q for illustration here) but another antecedently present variable: p in this instance. (The "antecedently present" was emphasized in Remark 0.2)). Here $\Diamond$ is defined as $\neg\Box\neg$; to get the two-variable form from the one-variable form by substituting $p \wedge q$ for p and using the fact that both $\Box$ and $\Diamond\Box$ provide monotone contexts in any normal modal logic, as does $\Diamond$ itself, and this is all true of arbitrary monotonic modal logics in fact, which bring with them further examples of consistent modal logics with isolating theorems, such as the monotonic logics extending EM by $\Diamond^m p$ (for any given $m \geq 1$). But again we can obtain corresponding non-isolating axioms $\Diamond^m\bot$ from these by substituting $\bot$ for p, and conversely, using the monotone nature of the contexts involved and the provability of $\bot \to p$, recover the original isolated form, establishing removability.[23]

The reference to monotonic modal logics extending EM by this or that axiom in the preceding paragraph could equally be put by speaking of congruential modal logics and the same axioms, since monotonicity implies congruentiality, and if a congruential logic contains the EM-provable $\Box(p \wedge q) \to \Box p$, then it is monotone.[24] Thus while

[20]Not for $m = 0$, since we are avoiding the case in which L is inconsistent, containing all formulas. (As usual, $\Box^m$ indicates the m-fold application of $\Box$.)

[21]What about the comment after Proposition 1.2 above to the effect that no sublogic of the 'trivial' modal logic KT!, the smallest modal logic containing $\Box p \to p$ and its converse, can have any theorems constructed using only connectives listed in Prop. 1.2 together with $\Box$ which are linear, yet the Verum system proves $\Box p$? The unsurprising answer is that the Verum system is not a sublogic of KT! (and neither are the other logics mentioned in the present paragraph).

[22]See Prior [46], p. 103, last paragraph; I am taking some liberties over the exact formulation here. Corrective remarks in Prior [47] do not affect the present point.

[23]The formulas $\Diamond^m\bot$, which would be inconsistent in the setting of normal modal logics, are considered as candidate axioms for *regular* modal logics in Segerberg [51], p. 201. Segerberg did not clearly distinguish monotonicity from regularity in [51], defining only the latter (on p. 12) and doing so by means of two conditions, confusingly calling the monotonicity condition, that $A \to B$ should be provable only if $\Box A \to \Box B$ is, (closure under) RR – for 'the rule of regularity'. In the same work, indeed in its title, Segerberg used 'classical' for (Makinson's) 'congruential', but reversed the choice in the later [52].

[24]The same goes for prefixing $\Box$ to the antecedent and consequent of any non-modal formula $A \to B$ for which the inference from A to B is classically archetypal (see for example Połacik and Humberstone [44] or Humberstone [29], Digression on p. 347); this includes the case with $A = p$ and $B = p \vee q$ mentioned (with the prefixed $\Box$'s) in the second paragraph after Prop. 1.2 above. Note that whether $A(q)$ is taken as $\Box(p \wedge q) \to \Box p$ or as $\Box p \to \Box(p \vee q)$ for axiomatizing EM as $\mathsf{E} \oplus A(q)$, the latter $A(q)$ isolates q and so we are dealing, to use a variable-dotting notational convention about to be introduced, with a case of EM being $\mathsf{E} \oplus A(\dot{q})$, with q presumably not removable here: there are no archetypal inferences from one formula

$\mathsf{EM} \oplus A$ is by definition – specifically Def. 1.3(*iii*) – the smallest congruential extension of EM containing the formula A, it is also for the smallest monotone logic containing A. One last notational convention will be helpful for distilling the question of interest towards which the above discussion is heading. For a formula A used as a context formula $A(p)$ in which p occurs at most once, we will convey this information by writing $A(p)$ as $A(\dot{p})$. Then we can ask:

P -Q Is $\mathsf{EM} \oplus A(\dot{p})$ always the same logic as $\mathsf{EM} \oplus A(\top)$ or $\mathsf{EM} \oplus A(\bot)$?

Since, if p does not occur at all in $A(\dot{p})$, $A(\top)$ and $A(\bot)$ both coincide with $A(\dot{p})$, we have a trivial affirmative answer: $\mathsf{EM} \oplus A(\dot{p}) = \mathsf{EM} \oplus A(\top) = \mathsf{EM} \oplus A(\bot)$. The case of interest is that in which p does occur in $A(\dot{p})$, in which case it occurs exactly once, and so the question asks whether such an isolated variable is always removable by substitution (Def. 0.1(*ii*)) and moreover, specifically by substitution of one of the constants $\top, \bot$. Well – it is not quite a *question* proper yet, but only what we have called a proto-question. An actual question to which an affirmative or negative answer might be returned, arises only on the resolution of an indeterminacy highlighted earlier: what are the Boolean primitives? Any functionally complete set of such connectives was permitted in the definition of a modal logic, but as we have seen, the notion of an isolating formula, here present only in the dot over the parenthetical "p", is sensitive to the way that issue is resolved.[25] In Section 2 we will look at one choice of primitive connectives and the answer to the concrete question emerging from Proto-Question above when that choice is made (Theorem 2.2), before turning to another for which the answer is different (Proposition 3.2). But for the remainder of this section, some words are in order on the reason for selecting the monotonic modal logics for discussion.

Monotonic as opposed to what? There is no reason for selecting monotone rather than antitonic modal logics, other than their greater familiarity as a landmark on the path from (arbitrary) congruential to (specifically) normal modal logics, the family of antitonic modal logics not lying on this path at all. But if the choice is between monotone and arbitrary congruential modal logics – replacing EM in the Proto-Question with E, that is – then there is every reason for preferring the narrow focus. Consider, for example, $A(\dot{p}) = \Box p$ against this weaker background. (A less dramatic example was given in note 24.) We can use the neighbourhood semantics for congruential modal logics, with its models $\langle W, N, V\rangle$, W a nonempty set, $N : W \longrightarrow \wp(\wp(W))$ and $V(p_i) \subseteq W$ for each i. Recall that truth for a formula A at a point $w \in W$ in such a model $\mathcal{M}$, say ($= \langle W, N, V\rangle$), $\mathcal{M} \models_x A$ is defined in the usual inductive manner, with the following clause governing $\Box$-compounds:

to another in the (at most) 1-variable fragment of classical logic. (That is, with any a one-premiss inference in this fragment, the premiss is refutable or the conclusion is provable, or the premiss and conclusion are equivalent. 'Presumably', because this is not, as it stands, a rigorous proof of the claim made, in the first place because not all removability is removability by substitution and secondly because we have not even shown that the isolated variable fails to be removable by substitution in this case.) Note that we have not defined isolation-related notion for (candidate non-zero-premiss) *rules* as opposed to *axioms*, and extending a basis for E to one for EM by the monotonicity rule $A \to B\,/\,\Box A \to \Box B$ is not under consideration here.

[25] Given a functionally complete set of Boolean primitives, the best known example of a (proto-)question in modal logic which receives a different answer depending on exactly what those primitives are is that discussed in Makinson [35], but by contrast with that case, the current indeterminacy survives when attention is restricted to congruential modal logics.

$$\mathcal{M} \models_w \Box B \text{ iff } ||B||^{\mathcal{M}} \in N(w),$$

where $||B||^{\mathcal{M}}$ is $\{x \in W \mid \mathcal{M} \models_x B\}$, and the right-hand side of this "iff" condition is often read as "The truth set (in $\mathcal{M}$) of B is one of the neighbourhoods of w. The $\langle W, N\rangle$ reduct of such a model $\langle W, N, V\rangle$ is called the frame of the model and $\langle W, N, V\rangle$ a model on that frame. A formula is *valid on* a frame just in case it is true at every point in every model on that frame, and the theorems of the smallest congruential logic E coincide with the formulas valid on every (neighbourhood) frame.[26] A formula *modally defines* a class of neighbourhood frames when it is valid on precisely the frames in that class. With these preliminaries aside, we can illustrate why turning the above Proto-Question into a concrete question not for EM but for E itself receives, on one choice of Boolean primitives we shall consider again in Section 2, an easy negative answer:

E 1.5 (*i*) Suppose the Boolean primitives are $\wedge, \vee, \to, \neg, \top, \bot$ and we ask whether for every p-isolating formula $A(\dot{p})$, $\mathsf{E} \oplus A(\dot{p})$ coincides with at least one of $\mathsf{E} \oplus A(\top)$, $\mathsf{E} \oplus A(\bot)$. We can return an immediate negative answer, which is actually independent of the choice of Boolean primitives, since we choose as $A(\dot{p})$ the formula $\Box p$, in which none occur. It is not hard to see that $\Box p$, $\Box\top$ and $\Box\bot$ modally define the classes of frames $\langle W, N\rangle$ satisfying conditions C_1, C_2 and C_3 respectively, understood as holding for all $s \in W$:

$$Y \in N(x), \text{for all } Y \subseteq W; \tag{C_1}$$

$$W \in N(x); \tag{C_2}$$

$$\varnothing \in N(x). \tag{C_3}$$

Thus, since a frame satisfying C_2, on which all the theorems of $\mathsf{E} \oplus \Box\top$ (alias $\mathsf{E} \oplus A(\top)$) are valid, need not satisfy C_1, and a frame satisfying C_3, on which all the theorems of $\mathsf{E} \oplus \Box\bot$ (alias $\mathsf{E} \oplus A(\bot)$) are valid, need not satisfy C_1, neither $\mathsf{E} \oplus A(\top)$ nor $\mathsf{E} \oplus A(\bot)$ coincides with $\mathsf{E} \oplus A(\dot{p})$, since neither of these logics contains $A(\dot{p})$.

(*ii*) For a variation on (*i*), consider the same choice of primitives as there, but with $A(\dot{p})$ taken as $\Box p \to \Box q$, asking whether $\mathsf{E} \oplus A(\dot{p})$ coincides with either of $\mathsf{E} \oplus A(\top)$, $\mathsf{E} \oplus A(\bot)$. This time there is a sensitivity to the particular choice of Boolean connectives taken as primitive, since if we had instead chosen the (again, functionally complete) set $\wedge, \leftrightarrow, \top, \bot$, with $B \to C$ defined as $B \leftrightarrow (B \wedge C)$, then the now non-linear formula $\Box p \to \Box q$ would not be one isolating the variable p. But for the announced primitives, which include $\to$, we have a contender for the $A(\dot{p})$ role, and can return a negative answer to the question raised by considering its substitution instances $A(\top)$ and $A(\bot)$. The smallest congruential modal logics containing these formulas, $\Box\top \to \Box q$ and $\Box\bot \to \Box q$, whose validity on a neighbourhood frame $\langle W, N\rangle$ requires only that for all

[26]Obviously this could equally well be put by saying that the theorems of E are exactly the formulas true at every point in every model, and this is how this result, conveniently proved in Chapter 9 of Chellas [6], is formulated, since Chellas does not isolate the concept of a frame – either a neighbourhood frame or, for that matter, a Kripke frame. For present purposes it is convenient to have the notion of validity on a frame available, as will be evident immediately. Some of the terminology differs from that used here: Chellas calls congruential modal logics classical (following a suggestion of Segerberg [51], retracted in Segerberg [52]) and Chellas calls neighbourhood models 'minimal models' (and Kripke models 'standard models'). A more recent overview of neighbourhood semantics is provided by Pacuit [41].

$x \in W$, $W \in N(x)$ should imply $Y \in N(x)$ for all $Y \subseteq W$, in the first case, or that for all $x \in W$, $\varnothing \in N(x)$ should imply $Y \in N(x)$ for all $Y \subseteq W$, in the second, conditions evidently not securing the validity of $\Box p \to \Box q$, which would require that for all $x \in W$, if $N(x) \neq \varnothing$ then $N(x) = \wp(W)$. (Of course there is a much simpler semantic description of this 'constant-valued' modal logic $\mathsf{E} \oplus \Box p \to \Box q$, namely as the set of formulas true on every Boolean valuation which interprets $\Box$ as the constant-true 1-ary truth-function and also on every Boolean valuation which interprets $\Box$ as the constant-false 1-ary truth-function.[27]) The modal logic just described, in which any two $\Box$-formulas are provably equivalent, is the weakest modal logic to be both monotonic and antitonic. To make this point even more generally than the present setting might suggest, suppose we take modal logics as substitution invariant consequence relations. Then (1) if any two formulas have a common consequence – for all A, B there is a C with $A \vdash C$ and $B \vdash C$, and equally well (2) if any two formulas are consequences of some formula – for all A, B there is a C with $C \vdash A$ and $C \vdash B$, for $\vdash$ monotonic and antitonic, we have $\Box A \vdash \Box B$ for arbitrary A, B. In the case of (1) for a given A, B we have $A \vdash C$ and $B \vdash C$ as promised by (1), so by monotonicity and the former, we have $\Box A \vdash \Box C$, and by antitonicity and the latter we, have $\Box C \vdash \Box B$, and thus combining these interim conclusions, we get $\Box A \vdash \Box B$. Similar reasoning applies in the case of (2) It might seem that we have something more general still, namely that we have $\Box A \vdash \Box B$ from the weaker hypothesis that each pair of formulas A, B have a common consequence or else a common formula of which is a consequence. But this is not more general after all, since we can take the case in which A, B are distinct propositional variables and appeal to substitution invariance to conclude that classification – monotone or antitone – for that case applies across the board.

(*iii*) As we started with the logic of noncontingency in Section 0 we may as well wheel in an example from that discussion here, since Δ is congruential in the logic determined by any class of frames, and we shall not bother to rewrite "Δ" for the sake of applying the above neighbourhood semantics verbatim, as we shall continue to think of it in terms of the Kripke (accessibility-relational) semantics. (One could of course pursue Δ-based modal logic using the neighbourhood semantics explicitly, and steps have been taken in this direction in Fan and Ditmarsch [15], and Fan [9], with a more extensive treatment in Fan [13].) The simplest example mentioned in Section 0 of an isolating formula was $\Delta\Delta p$, but here we can consider an even simpler example: Δp, axiomatizing the class of frames in which each point has at most one point accessible to it, and so providing the noncontingency version of the logic in the Chellas nomenclature as KD_c (D_c being the converse $\Diamond p \to \Box p$ of the well-known D axiom of deontic logic), itself having wide application in temporal and dynamic interpretations of $\Box$-based modal logic, as well as contrasting strikingly with the Verum logic already mentioned several times as the simplest consistent normal modal logic with an isolating axiom in the language with $\Box$ (or $\Diamond$) as primitive: instead of having no consistent proper extensions, KD_c has infinitely many of them. And the change of primitive modal notation from $\Box$ to Δ has in effect reduced the two occurrences of p in D_c to one, so that now we have

[27]Here a *Boolean valuation* is a bivalent assignment of truth-values to formulas which respects the conventional association of truth-functions with the Boolean connectives. This example would be described in the terminology of [26], esp. 3.24, as a case of hybridizing two Boolean connectives.

an isolating formula effecting the axiomatic extension. It remains only to check that we cannot replace that one occurrence by a variable-free formula to the same effect. But by induction on the complexity of variable-free formulas we see that every such formula, or else its negation, is already valid on every frame and is already in K^{Δ} (axiomatizations of which can be found in Section 4). Zolin [59], p. 541 observes that for any consistent extension $\mathsf{L} \supseteq \mathsf{KD}_c$, we have $\mathsf{L}^{\Delta} = \mathsf{K}^{\Delta} \oplus \Delta p$. This collapsing of many distinct $\Box$-logics to a single Δ-logic is a reflection of the notorious inability of the Δ-language to distinguish between the numbers 0 and 1: whereas a point validates every $\Box$-formula iff it has no successors (i.e., no points accessible to it), a point validates every Δ-formula iff it either has no successors or exactly one successor. (For the local notion of validity deployed here, see note 31.) ◀

2 Monotonic Modal Logics and Positive–Negative

To guide the discussion here, it will be helpful to go through some concepts and results pertaining to positive and negative occurrences of sentence letters, as they are presented in Blackburn et al. [4]. These authors choose as the primitive connectives for their modal object language (in the monomodal case which concerns us here) $\bot, \neg, \vee$, and $\Diamond$. Nothing in what they say about positive and negative occurrences depends on the choice of $\vee$ rather than $\wedge$ (or taking both as primitives) or on the choice of $\Diamond$ rather than $\Box$. But the absence of $\to$ and $\leftrightarrow$ from the primitives is important, more especially in the latter case (as is stressed in the discussion after Theorem 2.2 below). The sole significance of omitting $\to$ from the primitive connectives is that this makes possible a pleasantly simple (and indeed familiar) definition of positive and negative occurrences of sentence letters/propositional variables, given in [4] (p. 151) as Definition 3.34: An occurrence of a sentence letter p_i in a formula is a *positive* occurrence if it is in the scope of an even number of negation signs in that formula; it is *negative* if it is, instead, in the scope of an odd number of negation signs.[28] We can complicate things to allow for primitive $\to$ by simply saying that we apply this same definition to a given formula by looking not at the parity of occurrences of $\neg$ in whose scope p_i lies in the formula itself, but in the formula that results after replacing every subformula of the form $A \to B$ by $\neg A \vee B$ – in other words by simply treating the primitive $\to$ as defined for the sake of drawing the positive/negative distinction among occurrences of variables in this simple way.[29]

[28] I have written p_i where [4] has p here. The present positive/negative contrast is a version of the same contrast as applied to predicate letters on the first page of Lyndon [33].

[29] However, this way of drawing the positive/negative distinction by no means requires the classical equivalence of $A \to B$ with $\neg A \vee B$ and so is available for intuitionistic logic, for which one can similarly prove Lemma 2.1 below (with or with out an added monotone $\Box$). Further to this non-classical theme: discussing pure implicational formulas, though not taking $\to$ to be the material implication connective of classical logic, Anderson and Belnap [1], in fact talk about antecedent parts and consequent parts of a formula, rather than negative and positive parts – 'part' meaning subformula-occurrence – which, when the parts in question are sentence letter occurrences amounts to negative and positive occurrences by the present criterion. Of course, one could use this method to handle $\to$ as material implication in a functionally complete setting, rather than appealing to the paraphrase in terms of negation and disjunction, as one might well wish to if $\neg A$ itself is regarded as an abbreviation for $A \to \bot$: see Definition 9.9 on p. 105 of van Benthem [3]. As with Anderson and Belnap, Schütte [50], §1.5, speaks of positive and negative 'parts', but not taking these to be on the

We now look at a variation, appearing as Lemma 2.1 below, on Lemma 3.37 in Blackburn et al. [4], p. 152, which tells us that if all occurrences of p_i in a modal formula $A(p_i)$ are positive then $A(p_i)$ is a monotone context and if all occurrences of p_i are negative then $A(p_i)$ is antitone. Blackburn et al. ask the reader to prove both parts, 'positive ⇒ monotone' and 'negative ⇒ antitone', simultaneously by induction on the complexity of (= number of primitive connectives in) A, as Exercise 3.5.3 (p. 156). Rather than *monotone* and *antitone* the authors in fact say "upward monotone" and "downward monotone". In addition, they do not understand these terms as we have, instead taking them to express model-theoretic notions: (upward) monotone for $A(p_i)$ means that if $A(p_i)$ is true at a point in a Kripke model with valuation component V (assigning subsets of the universe of the model to the sentence letters) it remains true at that point in any model with valuation V' like V on each p_j ($j \neq i$) and such that $V(p_i) \subseteq V'(p_i)$. And similarly in the case of 'antitone' or "downward monotone" except that "$\subseteq$" is replaced by "$\supseteq$". The reason for this is that the application Blackburn et al.[30] make of the lemma in question is to the availability of first order conditions locally corresponding to modal formulas, rather than to the behaviour as monotone or antitone of contexts in different modal logics. A key part of the story remains pertinent to our own application (in the proof of Theorem 2.2): a variable all of whose occurrences in formula are positive can be replaced by $\bot$, and one all of whose occurrences are negative, by $\top$ and the replacement formula will be in a certain respect equivalent to the original. In Blackburn et al.'s discussion, the equivalence is a matter of validity at the same points[31] – a prelude to their presentation of Sahlqvist's Theorem – whereas, here, the relevant equivalence relation is a matter of equi-provability in a logic. Rewriting this lemma, then, to suit our version of the concepts involved gives us the following formulation of the result, a proof of which is given in the Appendix (Section 5), for anyone who might want it:

L 2.1 *Let* L *be any monotonic modal logic, with Boolean primitives as in Examples 1.5. Then*

(1) *If all occurrences of* p_i *in a formula* $A(p_i)$ *are positive, then* $A(p_i)$ *is monotone according to* L*, i.e., for all formulas* B, C*, if* $B \vdash_{\mathsf{L}} C$ *then* $A(B) \vdash_{\mathsf{L}} A(C)$*;*

(2) *If all occurrences of* p_i *in* $A(p_i)$ *are negative, then* $A(p_i)$ *is antitone according to* L*, i.e., for all formulas* B, C*, if* $B \vdash_{\mathsf{L}} C$ *then* $A(C) \vdash_{\mathsf{L}} A(B)$*.*

token ('occurrence') side of the type/token distinction, and calls a subformula A of B a positive part of B when A's truth (relative to any way of assigning truth values) guarantees B's truth, and a negative part of B when A's falsity guarantees B's truth. This, when parts are restricted to sentence letters, has nothing to do with the negative/positive occurrence contrast in play in the present discussion. On the present usage, the sole occurrence of p in $p \wedge q$ is positive rather than, as for Schütte, a negative part of $p \wedge q$; similarly on our usage here all three sentence letter occurrences in $p \wedge (q \vee r)$ are positive, whereas on Schütte's usage, the occurrence of p is a negative part of the formula, while q and r are neither positive nor negative parts of the formula (though $q \vee r$ is a negative part).

[30] As in van Benthem [3], p. 102, first new paragraph.

[31] Here, as in [26], p. 289, or [29], p. 185, we distinguish sharply between frame-based (valuation-independent) notions of *validity* and model-based notions of *truth*. Instead of saying that A is valid at $x \in W$ in the frame $\langle W, R\rangle$ (or: "x validates A in this frame") one can with equal legitimacy speak of validity on (or in) the 'pointed frame' $\langle W, R, x\rangle$, just as in the corresponding case with models.

It is perhaps worth explicitly mentioning that even if we forget entirely about modal logic and concentrate on the $\Box$-free formulas, this lemma tells us that for classical propositional logic with any primitives drawn from those listed positive contexts are monotone and negative contexts are antitone, where these references to contexts are to formulas $A(p_i)$ in which occurrences of p_i are all positive or else all negative.[32] Indeed, in that setting one has a much stronger result, with *monotone* and *antitone* replaced by *monotone with side-formulas* and *antitone with side-formulas*, respectively (to use the vocabulary introduced at the end of Remark 1.4(i)).

We are now in a position to observe that counterexamples like those mentioned in Examples 1.5 go away if "E" there is replaced by "EM", with the same Boolean primitives in play, returning an affirmative answer to the question emerging from our Proto-Question in Section 1 for that choice of primitives.

T 2.2 *Take the Boolean primitives as* $\wedge, \vee, \rightarrow, \neg, \top, \bot$. *Then for every p-isolating formula* $A(\dot{p})$, $\mathsf{EM} \oplus A(\dot{p})$ *coincides with at least one of* $\mathsf{EM} \oplus A(\top)$, $\mathsf{EM} \oplus A(\bot)$.

Proof. Since p only occurs once in any $A(\dot{p})$, all of its occurrences are positive or else all are negative, in which case by Lemma 2.1 $A(\dot{p})$, which we now write simply as $A(p)$, is monotone or antitone according to EM. In the former case $\mathsf{EM} \oplus A(p) = \mathsf{EM} \oplus A(\bot)$, since $A(\bot)$ is a substitution instance of $A(p)$, so $\mathsf{EM} \oplus A(\bot) \subseteq \mathsf{EM} \oplus A(p)$, while for the converse, since $\bot \vdash_{\mathsf{EM}} p$ and $A(p)$ is monotone, $A(\bot) \vdash_{\mathsf{EM}} A(p)$, so $\mathsf{E} \oplus A(p) \subseteq \mathsf{EM} \oplus A(\bot)$. A similar argument handles the case of $A(p)$ antitone, *mutatis mutandis*. ■

Of course, the removal by substitution of constants for isolated variables typically heralds considerable subsequent simplification, making use of the equivalence of $B \vee \top$, $B \wedge \top$, with $\top$, B, resp. (and $B \vee \bot$, $B \wedge \bot$ with B, $\bot$, resp.) and other such Boolean moves, so that, for example with $A(\dot{p})$ as $\Box(q \rightarrow \Diamond(q \rightarrow \Box(p \vee q)))$ we have, as $A(\bot)$ – the relevant case, since the isolated occurrence of p is positive – $\Box(q \rightarrow \Diamond(q \rightarrow \Box(\bot \vee q)))$ which then further simplifies to $\Box(q \rightarrow \Diamond(q \rightarrow \Box q))$ as an axiom yielding the same monotonic modal logic as the original isolating candidate axiom $A(\dot{p})$.[33]

R 2.3 We would have a similar result for the case antitonic modal logics, with a suitable reinterpretation of the terms in Lemma 2.1 and the same Boolean primitives as there. To adapt the reasoning for this case we need to count an occurrence of p_i as positive if it lies in the scope of an even number of occurrences of $\neg$ or $\Box$, and otherwise as negative. And as a provisional or partial explanation of the appearance of isolating formulas as axioms for the Δ-based examples of Section 0, the fact that no variant of Lemma 2.1 is available for the Δ language, Δ-contexts being neither antitone nor monotone. ◀

[32] In such formulas $A(p_i)$ (whether or not $\Box$ is present), p_i is said by Blackburn et al. ([4], p. 153) to occur *uniformly*.

[33] In normal modal logics we have further simplifications available in view of the equivalence of $\Box\top$ with $\top$ and in those among them extending KD, still further, with $\Box\bot$ equivalent to $\bot$.

In the foregoing discussion, we have put things in terms of extending arbitrary monotonic logics rather specifically extending EM, which we are entitled to do because the proof of Lemma 2.1 appeals only to what is provable in EM and not to what is not provable there, but in the following Corollary to Theorem 2.2, we put things again in terms of EM for the sake of connecting directly with the formulation of the Proto-Question in Section 1, which has now become a concrete question on fixing the stock of Boolean primitives as in our recent discussion:

C 2.4 *With the Boolean primitives listed in Theorem 2.2 for all $A(\dot{p})$, p is removable from $A(\dot{p})$ in the extension of* EM *to* $\mathsf{EM} \oplus A(\dot{p})$.

Now, there may be several isolated variables in $A(\dot{p})$ aside from p, so that after the removal of p we are still dealing with an isolating candidate axiom, $B(\dot{q})$, for example, in which case we must apply the removal procedure again, and thus, cutting a long story short by induction on the number of isolated variables in A in the envisaged extension of EM by A to see that all of them can be removed to give a non-isolating formula B with $\mathsf{EM} \oplus B = \mathsf{EM} \oplus A$. Note that this does not merely say that for any isolating formula A there exists a non-isolating formula B with $\mathsf{EM} \oplus B = \mathsf{EM} \oplus A$ – something already observed to be trivially the case, or at least, to be the case independently of the choice of either Boolean or modal primitives or base logic, since we can always take B as $A \wedge A$, to cite one of the illustrative options from the discussion in Section 0 before Definitions 0.1. Some terminology will help to bring this issue into sharper relief:

D 2.5 Let us say that the extension of L to $\mathsf{L}^+ \supseteq \mathsf{L}$ has the *isolation property* when there is some formula $A(\dot{p}_i)$ such that $\mathsf{L}^+ = \mathsf{L} \oplus A(\dot{p}_i)$ and p_i is not removable from $A(\dot{p}_i)$ in the extension of L to L^+ (as understood in Def. 0.1(i)). ◀

A more general definition would take into account the possibility that $\mathsf{L}^+ \supseteq \mathsf{L}$ is $\mathsf{L} \oplus \Gamma$ where $A(\dot{p}_i) \in \Gamma$ is a formula from which p_i is not removable in the extension in question, and does not occur in $\Gamma \smallsetminus \{A\}$. But the definition given suffices for present purposes. (A stronger version of the isolation property is tentatively introduced in Definition 2.9.)

R 2.6 The term *extension* here is being used in something other than its usual sense, not for the logic L^+ itself – an extension of L in that usual sense, and no doubt of many other logics as well, such as all the sublogics of L – but specifically for transition from L (not any of those other logics) to L^+: for the extending of L to L^+. (This is at least approximately what is sometimes called a process–product ambiguity.) Thus we may formally identify the extension, in this sense, of L to $\mathsf{L}^+ \supseteq \mathsf{L}$ with the ordered pair $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ and say that this 'extension pair' enjoys the isolation property precisely when $\mathsf{L}^+ = \mathsf{L} \oplus A(\dot{p}_i)$ with p_i is not removable from $A(\dot{p}_i)$. (We have written "$\supseteq$" rather than "$\supsetneq$," as in Def. 2.5; in the case in which $\mathsf{L}^+ = \mathsf{L} \oplus A(\dot{p}_i) = \mathsf{L}$, because already $A(\dot{p}_i) \in \mathsf{L}$, p_i is removable from $A(\dot{p}_i)$ in "extending" L to L^+ as we can trade in $A(p_i)$ for the p_i-free B taking B as $\top$, since the current L^+ – alias L – coincides with $\mathsf{L} \oplus B$.) ◀

In the more relaxed terminology of Definition 2.5 itself, Corollary 2.4 tells us that (relative to the choice of primitives listed there) the extension of EM to any $\mathsf{L}^+ \supseteq \mathsf{EM}$ lacks the isolation property. By contrast, Example 0.3 showed that the extension of K^Δ to $\mathsf{K}^\Delta \oplus \Delta\Delta p$ had the isolation property, as was also noted in Example 1.5(*iii*) for the extension of the same base logic to $\mathsf{K}^\Delta \oplus \Delta p$, with (*i*) and (*ii*) serving other cases where the extensions lacked the isolation property, checking which is made easier by the fact that the logics have been referred to by specifying the isolating formula that needs to be checked for the removability of its isolated variable(s). For example, in chapters 7 and 8 of Chellas [6], EM is officially introduced as (what we would call) $\mathsf{E} \oplus \mathsf{M}$ where M is the formula $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ and so not actually an isolating formula at all.[34] Each of the two sentence letters in such formulas is what is called in Definition 2.7(*ii*) *isolable* meaning that its number of occurrences can be reduced to 1 and yield the same logic (the same congruential extension of E), as is also the case with the more blatantly reduplicative ($A \wedge A$ etc.) non-isolating formulas. As just noted, though, this doesn't mean we can isolate both variables at once in a formula giving the same logic.

For thinking about the case of $\Box(p \wedge q) \to (\Box p \wedge \Box q)$, as well as for defining a strengthened version of the isolation property, it is helpful to introduce the idea of a variable being L-reduced in a formula, intended to capture the informal idea that it does not have gratuitously many occurrences in that formula as a candidate new axiom for extending the (for definiteness, congruential modal) logic L. Slightly less informally, The variable p_i is L-reduced in A if the number of occurrences of p_i cannot be decreased in any formula yielding the same extension of L as A without at the same time increasing the number of occurrences of some other variable. Putting it less 'negatively': if any formula L-equivalent to A with the same variables occurring in it as A with fewer occurrences of p_i must have more occurrences of some other variable. Whether this or some variation (for example, omitting (3) of Def. 2.7(*i*)) represents the best way of capturing the relevant idea is not entirely clear, so the following precise spelling out of the definition is proposed tentatively. In this formulation, we denote by $Occ(p_i, A)$ the number of occurrences of p_i in A. (An inductive definition could easily be given, but we have been taking this concept as understood throughout.) We take the opportunity to include a definition of the concept of isolability used with an informal gloss in part of our earlier discussion.

[34] In fact, Chellas uses the corresponding schema, with distinct schematic letters replacing the two sentence letters here, so what we have is a non-isolating schema in the sense of Definition 1.1(*ii*). Pacuit [41], p. 53, follows suit. The present author feels about this longer form the way that Montgomery and Routley felt ([39], p. 319) about one of the Lemmon–Gjertsen axioms for $\mathsf{S5}^\Delta$ when they adapted it for an axiomatization of KT^Δ, speaking of "weakening the unnecessarily strong first axiom" – though spelling out what strength and weakness amount to here in such a way as to make 'unnecessarily strong' formulas inappropriate as axioms would be no easy task. One reason Chellas may have had was a desire to avoid arbitrarily choosing between the two equally good shortenings of $\Box(p \wedge q) \to (\Box p \wedge \Box q)$, namely $\Box(p \wedge q) \to \Box p$ and $\Box(p \wedge q) \to \Box q$; informally put: the choice between two forms neither of which treats p, q in a symmetrical manner is forced by the fact that we can remove p and we can remove q but we cannot remove p and q from the symmetrical non-isolating formula. (A similar example arises in Remark 4.2 in which it is noted that we can remove r from a formula there called **Ku2′** constructed from p, q, r; we could equally well remove q, leaving r intact, but no formula constructed just from p could play the same axiomatic role.) Another (more likely) reason is so that Chellas could elegantly contrast the long form with its converse ('aggregation' or 'agglomeration') and consider the biconditional that combines the two and amounts, among congruential modal logics, to regularity (as defined at the end of Remark 1.4(*i*) above).

D 2.7 (i) A variable p_i is L-*reduced in A* iff there is no formula B such that

(1) $\mathsf{L} \oplus A = \mathsf{L} \oplus B$;

(2) for all k, $Occ(p_k, A) > 0 \iff Occ(p_k, B) > 0$;

(3) for all $j \neq i$, $Occ(p_j, B) \leq Occ(p_j, A)$;

(4) $Occ(p_i, B) < Occ(p_i, A)$.

(ii) A variable p_i is *isolable* in A (over L) just in case p_i is isolated in A or else there is a B satisfying (1) and (2) of (i) and this modified version of (4):

$$(4')\ 1 = Occ(p_i, B) < Occ(p_i, A),$$

and p_i is *cost-free isolable* in A (over L) if p_i is isolated in A or there is a B satisfying (1), (2) and (3) of (i) as well as (4′).

(iii) A formula A is L-*reduced* iff for each variable p_i occurring in A, p_i is L-reduced in A. ◀

Thus reducibility is a matter of lowering the number of occurrences of a propositional variable, in the case of interest, lowering it to 1, i.e. isolability.[35] Reducibility contrasts with removability (Def. 0.1(i)) which was a matter of being able to get rid of the variable altogether (in the notation above, lowering $Occ(p_i, A)$ to 0). For our current concerns, the former serves potentially to move from a non-isolating formula to an isolating formula by eliminating gratuitous occurrences (from $(A(q \wedge q)$ to $A(q) = A)\dot{q})$ for instance), and the latter to remove a solitary occurrence to move to an non-isolating formula from a spuriously isolating formula (from $A \wedge (q \vee A)$ to A, with A q-free, for instance).[36]

E 2.8 (Here we revisit some of the content of the paragraph after Remark 2.6 above.) Applying this to the case of $A = \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$, we see that p is not E-reduced in A, since omitting the first conjunct of the consequent would reduce $Occ(p, A)$ while yielding the same extension of E, and similarly, q is not E-reduced in A. Taking the case of q, and 'reducing' to $\Box(p \wedge q) \rightarrow \Box p$ we can ask whether p is isolable in this formula (over E), to which the answer is *yes* because this formula yields the same extension (namely EM) of E as the formula $\Box(p \wedge q) \rightarrow \Box q$, in which p is isolated, but p is not cost-free isolable in $\Box(p \wedge q) \rightarrow \Box p$ since (assuming, as in note 24, that there is no linear formula effecting this same extension of E) we trade in this formula for $\Box(p \wedge q) \rightarrow \Box q$ only at a 'cost' of one additional occurrence of q, violating (3) of Definition 2.7(i). ◀

Turning now to the mooted strengthening of the isolation property, we recall that this property is characterized in Definition 2.5 in terms of the *existence* of a isolated

[35]The latter, p_i's isolability in $A(p_i)$ over a logic L, amounting to the existence of a formula $B(\dot{p}_i)$ constructed from the same variables as $A(p_i)$, such that $\mathsf{L} \oplus A(p_i) = \mathsf{L} \oplus B(\dot{p}_i)$.

[36]These examples are chosen for simplicity and are misleading in one respect: they use the special case, with respect to an unspecified L, of provable equivalence in L, of the broader equivalence relation holding between formulas B, C when $\mathsf{L} \oplus B =, \mathsf{L} \oplus C$.

formula meeting a further ('unremovability') condition to block its being trivially possessed by every extension pair. The strengthening envisaged is, roughly speaking, to replace this existential condition with a universal condition, though because of the need, again, to avoid trivialising the property, we use the notion of reducibility just introduced (whence the 'roughly speaking' qualification). The shift from existential to universal quantification can be regarded as a shift from what in the discussion leading up to Def. 0.1 was called the "*availability* of candidate modal axioms in which some variable has a solitary occurrence" to that referring to the *unavoidability* of such axioms:

D 2.9 For any congruential modal logics $\mathsf{L}^+ \supseteq \mathsf{L}$, the extension pair $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ has the *strong isolation property* iff for every L-reduced A, if $\mathsf{L}^+ = \mathsf{L} \oplus A$, then A is an isolating formula. ◀

We show that the strong isolation property is at least as strong as its 'weak' namesake, using the following lemma, stated here without proof:

L 2.10 *For any formula A and any congruential modal logic L, there is an L-reduced formula A' with $\mathsf{L} \oplus A = \mathsf{L} \oplus A'$.*

P 2.11 *For all congruential modal logics L, L^+, for which $\mathsf{L}^+ = \mathsf{L} \oplus A$, if $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ has the strong isolation property, then $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ has the isolation property.*

Proof. Take $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ with the strong isolation property. Thus for every L-reduced A' with $\mathsf{L}^+ = \mathsf{L} \oplus A'$, A' is an isolating formula. By Lemma 2.10, there is some such L-reduced isolating A', which gives $\langle \mathsf{L}, \mathsf{L}^+ \rangle$ the isolation property. ■

All cases of the (what we are now calling) the isolation property cited in Section 0 as well as under Examples 1.5 – namely (in order) $\langle \mathsf{K}^\Delta, \mathsf{K}^\Delta \oplus \Delta\Delta p \rangle$, $\langle \mathsf{E}, \mathsf{E} \oplus \Box p \rangle$, $\langle \mathsf{E}, \mathsf{E} \oplus \Box p \to \Box q \rangle$, and $\langle \mathsf{K}^\Delta, \mathsf{K}^\Delta \oplus \Delta\Delta p \rangle$, in which the extending axiom is a linear formula, which suffices for the isolation property to coincide with the strong isolation property. In fact the author does not know for certain of any extension pair with the isolation property but not the strong isolation property; we return to the issue briefly in Section 4 (Remark 4.1).

As with the other concepts in play in this discussion, the part of Definition 2.5 that says "there is some formula $A(\dot{p}_i)$" meeting certain conditions reveals a tacit relativity, made explicit in the reformulation of Coro. 2.4, to the choice of Boolean primitives, since to denote $A(p_i)$ using this dot notation has us counting the number of occurrences of p_i in A, a number disturbed by reliance on non-linear (inter)definabilities on what might otherwise seem to be trivial changes in the primitives. This sensitivity to the primitive Boolean connectives persists with the concepts of an L-reduced formula and of the strong isolation property defined in terms of that notion. The following section looks at a representative case, mentioned already in Section 1: that of $\leftrightarrow$.

3 Changing the Boolean Primitives

In Proposition 1.2, reporting on the absence of linear tautologies, there appeared a familiar Boolean connective conspicuously missing from those we have been working with in (Examples 1.5 and) Lemma 2.1 and Theorem 2.2, namely the biconditional $\leftrightarrow$ – a contrast we could equally well be making by reference to exclusive disjunction, if-then-else ('conditioned disjunction') or any of many other perhaps less familiar Boolean connectives: essentially, all n-ary Boolean # for which $\#(p_1, \ldots, p_n)$ lacks any disjunctive normal form representation in which each p_i occurs uniformly in the sense of note 32.[37] The issue raised by these connectives is best introduced by noting that after defining an occurrence of a sentence letter in a formula to be positive (negative) if it lies in the scope of an even (resp. odd) number of negation signs, Blackburn et al. add parenthetically ([4], p. 151*f.*):

> This is one of the few places in the book where it is important to think in terms of the primitive connectives [recall that for [4], these do not include $\rightarrow$]. The occurrence of p in $\Diamond(p \rightarrow q)$ is negative, for this formula is short for $\Diamond(\neg p \vee q)$.

The fact that the definition of $\rightarrow$ employed is linear means that we can pair up the sentence letter occurrences in the $\rightarrow$-formulation with those in the formulation in primitive notation (and similarly with our own adaptation – see the discussion after Definitions 1.1 – of the positive/negative contrast when $\rightarrow$ is among the Boolean primitives). But there is no linear definition of $\leftrightarrow$ (to stick with this example) in terms of the Boolean primitives mentioned in Theorem 2.2; if there were, then the definition would provide a formula $A(\dot{p})$ equivalent to $p \leftrightarrow q$, in which, since p occurs only once $A(p)$ would be monotone or antitone, but this is not the case for $p \leftrightarrow q$.[38] This means that we would be hard pressed to the exhibited occurrence of p_i in a formula $p_i \leftrightarrow B$ as either positive or negative, in either classification would give a counterexample to Lemma 2.1 if $\leftrightarrow$ were added to the list of the primitives in play there. When $\leftrightarrow$ is spelled out in terms of those primitives we get two occurrences of p in $p \leftrightarrow q$ (for instance), one of them a positive occurrence and the other a negative occurrence (and similarly for q of course), exactly as happens with the nonlinear definition of Δ in terms of $\Box$ (or $\Diamond$) given in Section 0 and remarked on in this connection under Example 1.5(*iii*). Only now, the issue is raising its head not for the modal vocabulary but for the Boolean vocabulary.

We thus lose the route via Lemma 2.1 to Theorem 2.2, but that is not to say that there might be some other way of establishing a strengthening of Theorem 2.2 which permits the biconditional (or other similarly problematic connectives, such as exclusive disjunction) to be among the primitives in terms of which the $A(\dot{p})$ mentioned there merits that notation in featuring p only once. Let us settle the question, in Proposition 3.2, with what will turn out to be suitable counterexample, though first we need to take

[37] We could equally well say "conjunctive normal form" here. But is there any objective significance to the prominence afforded to $\wedge$, $\vee$ and $\neg$ on each of these choices? Might there be equally serviceable normal forms exploiting different functionally complete sets of Boolean primitives – material implication together with exclusive disjunction, for instance (to take an example from Pelletier and Martin [42])?

[38] This does not provide a monotone context for p since whereas $p \vee r$ follows from p, $(p \vee r) \leftrightarrow q$ does not follow from $p \leftrightarrow q$, and neither is the context antitone since $p \leftrightarrow q$ doesn't follow from $(p \vee r) \leftrightarrow q$ either. Here the reference to one thing following from another is to provable implication in classical propositional logic, since all of the modal logics under discussion here share this common non-modal core.

note of one feature of the isolating formula that will play the $A(\dot{p})$ role; note that we use the familiar label S4 rather than the explicit 'anatomical' label KT4, and drop the 'overdotting' in the course of the proof:

L 3.1 *Let $B(\dot{p})$ be the formula $\Box(q \leftrightarrow \Box p) \to (q \to \Box q)$. The normal extension of* KT *by $B(\dot{p})$ is* S4.

Proof. We get 4 in the form $\Box q \to \Box\Box q$ from $B(p)$ even against the backdrop of K, by substitution $\Box p$ for q in $B(p)$, and detaching the consequent, since the antecedent is now K-provable.

For the converse, we use an informal natural deduction argument. (Alternatively, just check that $B(p)$ is valid on every transitive reflexive frame.) Assume $\Box(q \leftrightarrow \Box p)$, with a view to deriving $q \to \Box q$. From this assumption, we get (a) $q \to \Box p$ (using T) and (b) $\Box(\Box p \to q)$, even in K, which (again in K) provably implies $\Box\Box p \to \Box q$ and hence, using 4, $\Box p \to \Box q$, which, with the help of (a), delivers the desired conclusion, $q \to \Box q$, as a truth-functional consequence. ■

Readers at home with natural deduction style modal reasoning should ignore this footnote.[39] Notice, in passing, that steps (a) and (b) in this proof extract the two oppositely directed conditionals from $\Box(q \leftrightarrow \Box p)$ at different modal depths, (a) discarding and (b) retaining the outer $\Box$. (The casual reference to extraction in this formulation should not of course be taken to suggest that the modally embedded $q \leftrightarrow \Box q$ is in fact the conjunction of $q \to \Box q$ with its converse, since the present point depends on taking "$\leftrightarrow$" as a primitive connective in its own right.) The formula $B(\dot{p})$ figuring in Lemma 3.1 is loosely inspired by the idea that $\Box$-formulas are 'special' in S4 in that, unlike arbitrary formulas, they provably imply their own necessitations.[40]

P 3.2 *If we add $\leftrightarrow$ to the Boolean primitives listed under Theorem 2.2 (whether or not we also remove some such primitives from the list), the resulting claim would be incorrect.*

Proof. By Lemma 3.1 KT $\oplus$ $B(\dot{p})$ is S4, where $B(\dot{p})$ is as in that lemma, so we need only observe that neither KT $\oplus$ $B(\top)$ nor KT $\oplus$ $B(\bot)$ coincides with S4. In fact KT $\oplus$ $B(\top)$ = KT $\oplus$ $B(\bot)$ = KT. To put this all directly in terms of (the $\leftrightarrow$-permitting variant of) Theorem 2.2, let $A(\dot{p})$ be the four-conjunct conjunction

$$((\Box q \wedge \Box r) \to \Box(q \wedge r)) \wedge \Box\top \wedge (\Box q \to q) \wedge\ B(\dot{p}),$$

[39] Those new to the area should note that in such informal natural deduction arguments as appear in the second half of the above proof, rules such as uniform substitution and necessitation under which the background logics are closed cannot be used in tracing out the consequences of assumptions, because the conclusions of such rules are not in general provably implied by their premisses in the logics concerned; it was in order to trigger the appropriate expectations in the main body of this paper – rules of *proof* figuring as closure conditions on logics rather than as rules of *inference* proper – that the phrase "candidate modal axioms" was chosen rather than (the coextensive term) "modal formulas" in our title. (Further discussion and references can be found in Humberstone [25], from which it will be evident that the rough description just given rather oversimplifies the situation.)

[40] See Observation 9.22.4 and the surrounding discussion in [26], p. 1305.

so we can redescribe the observation just made by saying that neither $\mathsf{EM} \oplus A(\top)$ nor $\mathsf{EM} \oplus A(\bot)$ coincides with $\mathsf{EM} \oplus A(\dot{p})$. (The first conjunct of $A(\dot{p})$ turns the monotonic EM regular, the second making it normal, and the third giving us the T of KT, the final conjunct returning us to the focus of Lemma 3.1.) ■

Although Proposition 3.2 answers (negatively) the concrete form of the Proto-Question from Section 1 for the enlarged set of Boolean connectives, it does not settle the issue of whether for all $A(\dot{p})$, p is removable from $A(\dot{p})$ in the extension of EM to $\mathsf{EM} \oplus A(\dot{p})$; all it does is to block a Theorem 2.2 style route to showing this. And in fact we already know that no other route is available, since according to Lemma 3.1, $\mathsf{KT} \oplus B(\dot{p}) =$ is $\mathsf{KT} \oplus \Box q \rightarrow \Box\Box q$, so this conjunct can be spliced into $A(p)$ in place of $B(p)$, showing q to be removable in the candidate axiom $A(\dot{p})$ for the extension $\langle \mathsf{EM}, \mathsf{S4} \rangle$. Thus we have the following

O Q Is there a formula $A(\dot{p})$ in the language with $\leftrightarrow$ among its Boolean primitives, for which the analogue of Corollary 2.4 fails for that language: p is not removable from $A(\dot{p})$ in the extension of EM to $\mathsf{EM} \oplus A(\dot{p})$?

The same question – essentially (for finitely axiomatizable extensions of EM), as to whether any of them have the isolation property in the language which adds primitive $\leftrightarrow$ – can be asked more generally for any other additional Boolean primitives. But for the remainder of this section, we focus on an aspect of the material biconditional which seems particularly relevant given the prevalence noted in Section 0 of isolated variables in proposed axioms for systems of noncontingency-based modal logics.

One can imagine various grounds that might be offered for not treating the biconditional as a primitive connective. Extensive empirical research might reveal that no human language expresses "if and only if" monomorphemically, for instance.[41] But such parochial findings (if they were forthcoming) would have bear no logical significance, whatever their psychological interest might be. More systematic considerations are proposed by Hartley Slater at p. 190*f.* of [55]:

> Here we come to notice the always-evident fact that there are three natural deduction rules for every basic logical symbol in classical propositional logic. There is not just one elimination rule for 'and', but two [...] It is not the case, of course, that there are three rules in the case of 'if... and only if ..., for instance, or 'if ... then ... else ...'. But these can be defined in terms of the elementary ones.

But quite how, though mentioning conjunction and (in the ellipsed part of the passage quoted) disjunction, Slater would conjure up three rules instead of just a single introduction and a single elimination rule (alias Conditional Proof and Modus Ponens) for

[41]Gazdar and Pullum seem to have some such criterion in mind in [20], when speculating about truth-functional connectives in humanly possible languages, but in any case propose several semantic conditions one of which rules this out: namely that the associated truth-function should not have be one which value T when all its arguments have the value F. This rules out *neither/nor* too, concerning which the authors (p. 231) content themselves with suggesting that this connective involves the (e.g., transformational) incorporation of a negative element rather than being fundamental. A similar suggestion was made in Borowski [5] in response to an earlier paper by K. Halbasch making Reichenbach-like moves – see note 45 – using multigrade *neither/nor* constructions rather than exclusive disjunction.

plain old ‘if then’ is not clear. And in any case the most obvious natural deduction treatment of ‘iff’ *would* have – the magic number – *three* rules anyway: one introduction rule, a double-barreled variant of Conditional Proof, and two elimination rules, the two Modus Ponens style rules for the two directions of “$\leftrightarrow$”; so this particular line of thought is going nowhere.[42]

Rather than dismissing the biconditional as some kind of second-class citizen of the world of connectives, or, in the Boolean case, the conventionally associated truth-functions, let us compare it with our noncontingency connective Δ (which itself may have its own detractors as a primitive on grounds similar to some of those urged as showing $\leftrightarrow$ to be somehow ‘essentially derivative’). The intermediate case of a quantifier rather than a modal operator provides a pointer to the similarity involved here. But what quantifier? Here is Tharp ([56], p. 700) suggesting that nothing comes to mind. His notation has been left intact rather than being adjusted to match that of the present discussion:

> The standard quantifiers $\forall$ and $\exists$ can, for many purposes, be regarded as extensions of conjunction and disjunction. Consider the attempt to extend another connective, the biconditional, which is also commutative and associative. Let t be an arbitrary (infinite) truth assignment to the letters $P_0, P_1, P_2, \ldots$. Then t may be extended to assign a truth value $t(\phi)$ to each sentential formula ϕ. In an obvious sense the limits $\lim_{n\to\infty} t(P_0 \wedge P_1 \wedge \cdots \wedge P_n)$ and $\lim_{n\to\infty} t(P_0 \vee P_1 \vee \cdots \vee P_n)$ exist, but for certain t (e.g., if $t(P_i)$ is false for all i) $\lim_{n\to\infty} t(P_0 \leftrightarrow P_1 \leftrightarrow \ldots \leftrightarrow P_n)$ and does not exist. The fact that no quantifier suggests itself as a natural extension of $\leftrightarrow$ seems to be related to this discontinuity.

The interest of Tharp’s subsequent discussion notwithstanding, the comment that “no quantifier suggests itself” as a natural extension – or at least a natural analogue – of $\leftrightarrow$ seems rather swift. It is prompted by the idea that for a given n, a suitable version of the biconditional is given by (reverting to our own use of schematic letters) $A_1 \leftrightarrow \ldots \leftrightarrow A_n$, in which we follow Tharp in exploiting the associativity of $\leftrightarrow$ (in classical logic) in order to suppress parentheses, in the same way that the $n-1$-fold iteration of $\wedge$ and $\vee$ give what are felt to be natural n-ary incarnations of conjunction and (inclusive disjunction), and indeed, waiving the constraint that a function has a fixed number of arguments, to constitute, collectively, multigrade connectives (for combining any $n \geq 2$ formulas). This claim of what seems natural in the conjunction and disjunction cases is indeed an empirical psychological claim, for which support comes from the observation of the syntax of coordination and from the use of multigrade connectives in formal projects aimed at capturing this aspect of natural language syntax and ordinary reasoning.[43] This is why, in informal mathematical writing, one sees such things as the following, where the “(1)”, “(2)”, etc. abbreviate or label previously cited statements (and we use & and $\Leftrightarrow$ for what in the binary case would be the usual conjunction and

[42] Another discussion – exploratory rather than purportedly decisive – of relative fundamentality of among truth-functions appears in p. 217*f.* of Sider [54]. As with Slater, this is supposed to be an objective non-psychological (‘joint-carving’) matter. (In fact the discussion concerns connectives rather than the truth-functions conventionally associated with them, with an assumed background of classical logic. And Sider extends the discussion to predicate logic: “Similarly, which quantifier carves at the joints, $\forall$ or $\exists$?”)

[43] See §5 of McCawley [36], including p. 520 for the relevant work of J. R. Ross, as well as §3.5 of McCawley [38] and McCawley [37].

the biconditional):

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4),$$

in place of (1) $\Leftrightarrow$ (2) & (2) $\Leftrightarrow$ (3) & (3) $\Leftrightarrow$ (4). As is mentioned in Humberstone [26], p. 1136, there is a similar usage with $\Rightarrow$,[44] though here let us add that the inset formulation could instead be thought of as the application of a multigrade connective which combined with n statements to make a compound which is true just in case all of the components have the same truth-value. As is well known, this is not the effect obtained by iteration of the binary biconditional, which for example, yields from three components a compound not only under that condition but also, for instance, when the first component is true and the second and third are false.[45]

The incarnations as n-ary connectives of the multigrade biconditional (the "bi" no longer seem as apt as for the binary case), or rather the associated truth-functions, are paradigm cases of membership in Post's 'alternating' truth-functions.[46] But the feature just noted of the n-ary biconditional-style truth-functions is not shared with other alternating (linear, counting) truth-functions and where $\equiv$ is the multigrade connective whose incarnation ("$\equiv_n$", say) at n expresses that truth-function shows a conspicuous similarity with the noncontingency operator Δ, writing $v_x(A)$ for the truth-value of A

[44] The biconditional version is noted in Reichenbach [48], second new paragraph on p. 46; readers dipping into this discussion should note that Reichenbach uses a dot for conjunction and when he writes $\wedge$, he means exclusive disjunction.

[45] This is a well-known step in the proof of Leśniewski's theorem on when formulas constructed using only $\leftrightarrow$ are tautologous – see [26] Observation 7.31.6 – and from Reichenbach's similar observation in the case of binary exclusive disjunction, Reichenbach having explicitly emphasized the contrast in the case of exclusive disjunction between the iterated binary case and with the usual "exactly one is true" interpretation of the natural language construction in which with more than two disjuncts appear. Detailed discussion and references can be found on p. 783 of [26]. Gazdar and Pullum call disjunction so construed *generalized exclusive disjunction* ([20], p. 230); note that by contrast with the specifically binary case, this does not in general coincide with the negation of the 'generalized biconditional' (this latter making a compound that is true just when all components agree in truth-value). At this point Gazdar and Pullum disagree with a criticism made by the present author in reviewing (the collection containing) McCawley [36], saying that the criticism in question requires that generalized exclusive disjunctions have the truth-conditions just sketched, as opposed to being true just in case a proper subset of the components are true. (This latter amounts to the negation of the generalized biconditional compound of those same components.) Gazdar and Pullum (p. 230) write: "To the best of our knowledge there is no empirical evidence for this assumption [[i.e., the "exactly-one-true-component" account]]." Well, since this was all part of a criticism of McCawley's suggestion that generalized exclusive disjunction was a set-taking connective, the fact that the account in question was that endorsed by McCawley – "The use of the English word *or* that most closely matches the logician's 'exclusive *or*' yields a true sentence if and only if one of the conjuncts [[*sc.* components]] is true" ([36], p. 519) – would seem to have some bearing on the appropriateness of the criticism (urging, to put it in current terms, that "set-taking" be replaced by "multiset-taking"). But such *ad hominem* considerations aside, the 'assumption' in question is one available to any speaker of English, who will realise that when a heavy exclusivity-indicating stress is evident in such cases as "The box under the Christmas tree with your name on it will contain a new pair of skates OR a new phone OR the last outfit you told your mother you wanted," the take-home interpretation is that exactly one of these is on offer, rather than that some but not all of them are. (None of this should be taken to imply that the present author believes there is a separate exclusive sense of *or* in English.)

[46] These are also sometimes called *linear* or *counting* truth-functions, The "counting" nomenclature is used in Pelletier and Martin [42], *q.v.* for historical references; there seems to be some difference of opinion as to exactly what Post had in mind as defining membership in this class: see note 24 in French and Humberstone [19]. Note also that while not etymologically unrelated, this usage of *linear* is not the same as that from Definition 1.1(*i*), $p \wedge q$ is a linear formula, while the truth-function associated with $\wedge$ is not a linear function.

at x in some arbitrarily selected model, and representing the multigrade connective as taking a set X of formulas to a formula:

$v_x(\Delta A) = T$ just in case for all $y, z \in R(x)$, $v_y(A) = v_z(A)$;

$v_x(\equiv(X)) = T$ just in case for all $B, C \in X$, $v_x(B) = v_x(C)$.

One upshot of this similarity is that just as the condition on the right in the first case is automatically – one might be inclined to say 'vacuously' – satisfied when $|R(x)| < 2$, giving rise to the 0-1-insensitivity of noncontingency logic mentioned at the end of Example 1.5(*iii*),[47] so in the second case the condition on the right is automatically satisfied when $|X| < 2$, prompting the parenthetical restriction on multigrade connectives above as "combining any $n \geq 2$ formulas," lest anyone find this consequence intolerable.[48] With this analogy in mind, one might react to Tharp's suggestion that no quantifier comes to mind corresponding to the biconditional, would be puzzlement: what about the all-or-none quantifier of Thomason and Leblanc [57]? This is of course just the quantificational analogue of noncontingency (as was mentioned in note 2 of [28]). Since the latter amounts to the truth of the formula after the "Δ" at all or else at none of the points accessible to the point at which we are evaluating the Δ-formula. And this condition itself is naturally expressible in a biconditional formulation, taking x to be the point of evaluation we want: for all $y, z \in R(x)$ A is true at y if and only if A is true at z. Similarly, with $\equiv X$ above we can replace the right-hand side given there with "for all $B, C \in X$, $v_x(B) = T$ iff $v_x(C) = T$." At this juncture, an objection may be raised to the effect that this reveals no great affinity between noncontingency and material equivalence, since the "iff" can be replaced equally well by "only if" (or for that matter by "if") without any substantive change to the truth conditions on offer. Reply: but this is so also in the $\equiv X$ case, leaving the objector to wonder how compelling the generalization of binary $\leftrightarrow$ should be taken to be – but inviting the further reply that even just fixing on the original binary case we can specialize the $\equiv X$ treatment and get $v(A \leftrightarrow B) = T$ just in case for all $C, D \in \{A, B\}$, $v(C) = T$ only if $v(D) = T$. Clearly the biconditionality is being achieved by the use of the one-way metalinguistic condition because the quantifier prefix allows to first to pick A, B as respectively instantiating the variables C, D, and then to pick B, A as instantiating them, giving us both of the desired implications. (It also throws in the automatically satisfied case in which A is chosen both as C and D, and likewise when B is – cases which make no difference to the satisfaction of the condition as a whole.)

From the perspective of the question of the removability of isolated variables from candidate axioms, primitive Δ and $\leftrightarrow$ raise the same issue, namely their non-monotonicity (or more accurately, the fact that they are contexts which are neither monotone nor antitone), so that an isolated variable cannot be replaced in such contexts with a constant – at least not with a familiar $\bot$ or $\top$. The contexts $A(p) = p \leftrightarrow q$ (more generally: $\equiv(q_1, \ldots, q_n, p)$) and $B(p) = \Delta p$ and can be written (given sufficient expressive resources) as disjunctions of a monotone with an antitone context $(p \wedge q) \vee (\neg p \wedge \neg q)$

[47] Notation as in note 6.

[48] Another point of interest is that we cannot always simulate multigrade connectives with the use of sets of formulas in this way. Arguably this is the case for multigrade exclusive disjunction as construed above requires that if the several components are reified into a single entity to which the (let's still say) connective applies, that entity should be a multiset rather than a set. McCawley discusses this issue at p. 197*f.* of [38].

(more generally $(p \wedge q_1 \wedge \ldots \wedge q_n) \vee (\neg p \wedge \neg q_1 \wedge \ldots \wedge \neg q_n)$) and $\Box p \vee \Box \neg p$, which renders them 'co-convex' as it is put in Section 4,[49] though we conclude the present discussion by touching on a candidate Δ axiom there called **Zo3** (from an axiomatization of K^{Δ} given in Zolin [59]), here presented with schematic letters rather than propositional variables:

$$\Delta(A \leftrightarrow B) \rightarrow (\Delta A \leftrightarrow \Delta B).$$

This provides us with a good opportunity to illustrate the affinity between $\leftrightarrow$ and Δ highlighting a particular equational property of the truth-function satisfied by (the truth-function associated with) $\leftrightarrow$, namely the Medial Law: for all a, b, c, d, $(ab)(cd) = (ac)(bd)$, here representing the binary operation concerned by juxtaposition.[50] Suppose that an instance of Zolin's schema above has its antecedent true at a point x in some model, which means that for all y, z accessible to x we have (suppressing reference to the model), and writing the metalinguistic material biconditional as $\Leftrightarrow$:

$$(\models_y A \leftrightarrow B) \;\Leftrightarrow\; (\models_z A \leftrightarrow B),$$

and so, spelling this out,

$$(\models_y A \;\Leftrightarrow\; \models_y B) \;\Leftrightarrow\; (\models_z A \;\Leftrightarrow\; \models_z B).$$

By the Medial Law, this is equivalent to:

$$(\models_y A \;\Leftrightarrow\; \models_z A) \;\Leftrightarrow\; (\models_y B \;\Leftrightarrow\; \models_z B).$$

Recalling that we have all this for a particular – though arbitrarily selected – $y, z \in R(x)$, the quantificational principle that $\forall y, z(\phi(y, z) \leftrightarrow \psi(y, z))$ implies $\forall y, z(\phi(y, z)) \leftrightarrow \forall y, z(\psi(y, z))$, we conclude that x verifies the consequent, $\Delta A \leftrightarrow \Delta B$ of Zolin's axiom at x in this arbitrarily selected model, and hence that every instance of the schema in question is valid on all frames. That is obvious enough on inspection, but is worth seeing spelled out in detail for a fuller appreciation of the relation between the two non-monotone connectives, $\leftrightarrow$ and Δ, whose interaction we have been putting under the spotlight.

4 Coda on Noncontingency

The discussion in this section presumes, at times, a greater familiarity with modal logic than has been the case for the main body of this paper – in particular, toward the end, a familiarity with the general idea of canonical model completeness proofs. The publications cited in Section 0 on the extension of K^{Δ} by various candidate axioms mentioned the isolating formulas $\Delta 4$, $\Delta 5$ and ΔB in that capacity, as well as non-isolating variants

[49] $A(p)$ here can also be written as the *conjunction* of a monotone with an antitone context, as $(q \rightarrow p) \wedge (p \rightarrow q)$, which makes it also convex, to use again the terminology of Section 4 below.

[50] This is an easy consequence of associativity and commutativity for the operation in question and plays a salient role in giving $\leftrightarrow$ some of its distinctive classical behaviour, as illustrated in pp. 1131–1133 of [26]. We could have chosen the version of Zo3 which just uses $\rightarrow$ in place of $\leftrightarrow$, appearing in Section 4 as **LG2** (and as **MRb2**) but have opted for the present 'unnecessarily strong' **Zo3** to see as many occurrences of $\leftrightarrow$ as possible.

$w\Delta 4$ and $w\Delta 5$ which could replace them when attention was restricted to reflexive frames, i.e., in extensions of KT^{Δ}. Nothing was said, however, about how to axiomatize that logic, because in those publications (starting, in effect, with Montgomery and Routley [39]) what one encounters as the $\Delta\mathsf{T}$ for which $\mathsf{KT}^{\Delta} = \mathsf{K}^{\Delta} \oplus \Delta\mathsf{T}$ is:

$$p \to (\Delta(p \to q) \to (\Delta p \to \Delta q)),$$

a non-isolating formula which prefixes a "p" antecedent to the familiar K axiom (with $\Box$ replaced by Δ), first seen in Montgomery and Routley [39].

The weakest ($\Box$-based) normal modal logic considered in Montgomery and Routley [39] is KT, since they show that the various axiomatization they offer of this logic and some of its extensions capture exactly the Δ fragments by showing syntactically that the Δ-based logics are definitionally equivalent to their $\Box$-based cousins. (Thus a semantic completeness proof is not called for. As noted in Section 0, they start at KT so as to have the "$A \wedge \Delta A$" definition of $\Box A$ available.) In the case of KT, [39] offers three axiomatizations of (what we are calling) KT^{Δ}, of which two are worth recalling here. We shall call them **MRa** and **MRb**:

For **MRa**, as with all the axiomatizations to follow, we make additions to any basis for classical propositional logic including Modus Ponens as a rule, the modal rule taking us from A to ΔA,[51] and the following modal axioms:

MRa1 $\quad \Delta p \leftrightarrow \Delta\neg p$

MRa2 $\quad p \to (\Delta(p \to q) \to (\Delta p \to \Delta q)) \quad (= \Delta\mathsf{T})$

The $\leftrightarrow$ in **MRa1** here (as well as in **MRb2**, **LG1**, **LG2**, **Zo1**, **Zo3**, **Fa1** and the congruentiality rule (**RE**) below), let us say for definiteness, is to be regarded as a defined connective, the Boolean primitives for the present discussion being those given in Lemma 2.1. (Not that this affects our discussion, since 'isolated variables' issues are raised only for axioms not on this list.) Notice that neither of the (modal) axioms is an isolating formula, as were those ($\Delta 4$ etc.) initially sampled in Section 0. We return to this in Remark 4.1, which should be skipped by those wanting to get more expeditiously to what they may feel is the less trivial aspect of Δ-based modal logic: the logics not extending KT^{Δ}, in which (the invisible, underlying) $\Box$ is not in general definable. Indeed, such readers are advised to pass straight to the paragraph below in which axioms **Ku1**–**Ku3** appear.

We turn now to the second of the two Montgomery–Routley axiomatizations of KT^{Δ} to be mentioned here – which is the third to be found in [39], the second one there again providing a non-isolating basis – we have, by contrast with **MRa**, the typical variable-isolating feature of noncontingency bases:

For **MRb** we have the same rules as **MRa**, and the following three axioms, in the third of which we have highlighted the isolated variable by double underlining, as in Section 0:

[51] Similarly implicit is the rule of Uniform Substitution, which is to taken as part of all the axiomatizations considered here. Note that this rule and all rules to be described as modal rules here are rules of proof rather than rules of inference, in the terminology of note 39.

MRb1 $\Delta p \to \Delta\neg p$

MRb2 $\Delta(p \leftrightarrow q) \to (\Delta p \to \Delta q)$

MRb3 $\neg p \to (\Delta p \to \Delta(p \to \underline{\underline{q}}))$

Let us interrupt this survey of axiomatizations, to touch on the contrast between the isolation property and the strong isolation property (Def. 2.9).[52] This concerns the yet-to-be-seen-axiomatized K^{Δ} – but the details of how to axiomatize this logic do not matter to the present illustration since it involves the extension pair $\langle \mathsf{K}^{\Delta}, \mathsf{KT}^{\Delta} \rangle$ and so pertains to getting *from* the antecedently given K^{Δ} *to* a proper extension thereof. To return to the survey of axiomatizations, omit following numbered remark.

R 4.1 Apropos of the **MRa** axiomatization above it was noted that neither of the (modal) axioms was an isolating formula, by contrast with $\Delta 4$ etc. One might add further that the axioms, and in particular **MRa2**, alias $\Delta\mathsf{T}$, does not have the appearance of being obtained by reformulations of isolating axioms by artificial repetitions or other spurious insertions of what would otherwise be an isolated variable. In fact it is not hard to see, via Montgomery and Routley's syntactic observations and the fact that **MRa1**, **MRb1** and **MRb2** are all in K^{Δ}, being valid on all frames.[53]

$$(i)\ \mathsf{KT}^{\Delta} = \mathsf{K}^{\Delta} \oplus \Delta\mathsf{T} \text{ and } (ii)\ \mathsf{KT}^{\Delta} = \mathsf{K}^{\Delta} \oplus \textbf{MRb3}.$$

So this, given the already remarked on naturalness of the **MRa** axiomatization raises the possibility that the extension pair $\langle \mathsf{K}^{\Delta}, \mathsf{KT}^{\Delta} \rangle$ has the isolation property but lacks the strong isolation property. We can almost say that we know that it has the isolation property because of (*ii*) and the fact that **MRb3** is an isolating formula, but we should also check that the isolated variable is not removable (for this extension), which would certainly follow if we had a proof of the plausible seeming

KT^{Δ} C : for all formulas A such that $\mathsf{KT}^{\Delta} = \mathsf{K}^{\Delta} \oplus A$, A is constructed using occurrences of at least two propositional variables.

But the problems for the current would-be illustration of the isolation property without the strong isolation property, really arise with showing that $\langle \mathsf{K}^{\Delta}, \mathsf{KT}^{\Delta} \rangle$ lacks the strong isolation property. If we were to show this using $\Delta\mathsf{T}$, we would similarly need not only (i) and the fact that $\Delta\mathsf{T}$ is not an isolating formula, but also that it is a K^{Δ}-reduced formula, in the sense of Definition 2.7(*i*): there must be no alternative to it in the same variables, yielding the same extension of K^{Δ}, reducing the number of occurrences of any variable without increasing the number of occurrences of another. But **MRb3** is itself just such an alternative, reducing q's occurrences to 1 (so q is isolable in **MRb3** over K^{Δ}) while keeping p's occurrences at 3. This leaves several lines of inquiry open.

[52]The inclusion of this admittedly inconclusive discussion was prompted by a question raised by Jie Fan in his comments on an earlier draft of the present paper.

[53]Alternatively, for a lower reliance on Montgomery and Routley's syntactical observations and on the semantic completeness of the axiomatizations of K^{Δ} to be sampled below, for (*i*) see Theorem 5.9 in [17], and for (*ii*), see Proposition 4.3 below, which makes an analogous claim concerning the formula $(\Delta p \wedge p) \to \Delta(p \vee q)$, of which **MRb3** is a mild reformulation – substitute $\neg p$ for p, remembering that Δ and $\Delta\neg$ are equivalent and making a few truth-functional manipulations. "Mild reformulation" can be made precise in the following way: the two formulas give the same extension of the weak logic E^{Δ} mentioned below in this Remark (and in note 62).

Perhaps $\langle \mathsf{K}^{\Delta}, \mathsf{KT}^{\Delta}\rangle$ does after all lack the strong isolation property even though $\Delta\mathsf{T}$ cannot be used to show this. Perhaps we could weaken the starting point for the extension, and show that $\langle \mathsf{E}^{\Delta}, \mathsf{E}^{\Delta} \oplus \Delta\mathsf{T}\rangle$ has the isolation property but not the strong isolation property, for instance. (See note 62 below for a description of E^{Δ}.) Perhaps the definition of the strong isolation property in terms of reduced formulas needs tweaking to do justice to the informal idea of gratuitously reduplicating variable occurrences. The author has not investigated these options in any detail. The same goes for the 'isolation property' status – the plain isolation property, the strong version, or neither – of various extensions using the non-isolating axioms $w\Delta\mathsf{4}$ and $w\Delta\mathsf{5}$ mentioned in Section 0 in connection with p. 86 of Fan et al. [17], $\langle \mathsf{L}, \mathsf{L} \oplus A\rangle$ where A is one of those axioms and L is K^{Δ} or KT^{Δ}. ◀

Montgomery and Routley, to return to their [39] (though [40] is also relevant), give some further ∇-based axiomatizations of KT (which they seem to feel are sufficiently different from the Δ-based versions to be worth presenting separately), one of which they describe as obtained from the axiomatization by Lemmon–Gjertsen axiomatization of $\mathsf{S5}^{\Delta}$ (see note 7) by omitting its final axiom (as well as "weakening the unnecessarily strong first axiom" – see note 34). Let us turn to that axiomatization. It uses the same non-modal basis above and the A / ΔA rule, with the following axioms, listed in the original Polish notation on the right, with "Q" in place of ∇ and no special symbol for Δ, for which reason slight liberties have been taken with the de-Polonization – especially, rendering Lemmon and Gjertsen's "NQ" as "Δ" rather than "$\neg\nabla$" – to avoid cumbersome formulations:

LG1	$\Delta p \leftrightarrow \Delta\neg p$	*EQpQNp*	literally: $\nabla p \leftrightarrow \nabla\neg p$
LG2	$\Delta(p \leftrightarrow q) \to (\Delta p \to \Delta q)$	*CNQEpqCQpQq*	$\Delta(p \leftrightarrow q) \to (\nabla p \to \nabla q)$
LG3	$\Delta(p \to \underline{\underline{q}}) \vee (\Delta p \to p)$	*CQCpqCNQpp*	
LG4	$\Delta(p \to \nabla p)$	*NQCpQp*	

In place of **LG4**, we could equally well have: $\Delta(\Delta p \to p)$, since one can substitute $\neg p$ for p in **LG4** and contrapose, appealing to congruentiality for various simplifications (the rule (**RE**) being derivable from this basis in the way explained for **Zo1**–**Zo3** below) and all these steps are (essentially) reversible. This alternative form of **LG4** is one of the formulas attended to in the (modal definability) Theorem 5.1 in Zolin [59], mentioned above (note 6) apropos of one of the others ($\Delta\Delta p$) treated there. The q-isolating axiom **MRb3** is a purely truth-functional reformulation of **LG3**. It seems unlikely that q is removable from this formula as extending the logic axiomatized by the remainder of the Lemmon–Gjertsen basis to $\mathsf{S5}^{\Delta}$, or that q is removable from the isolating axiom in second Montgomery–Routley basis as extending to KT^{Δ} the logic axiomatized on deleting that axiom. It would be nice to be able to offer proofs of these conjectures, though.

We turn now to axiomatizations of K^{Δ}, which as we shall see, typically consist of three axioms (to be added to a suitable basis for classical propositional logic) one of which is an isolating formula, and in that formula two variables are isolated. Something approximating to an explanation of this is offered below, beginning in the paragraph in which a formula labelled "**Ku3**′" appears. We begin with Kuhn [32], where the axioms appear on p. 231.[54] There the axioms are taken as the instances of three axiom schemes listed, but for the sake of continuity with most of the discussion above we replace them with the corresponding representative instances, i.e., putting distinct propositional variables in place of distinct schematic letters and add a rule of Uniform Substitution to Modus Ponens to give the two non-modal rules.[55] All axiomatizations described here take as additional non-modal axioms all classical tautologies (or some non-modal basis sufficient to yield them) with the aid of Modus Ponens and Uniform Substitution. These three axioms correspond to Kuhn's schemata **A1**, **A3**, **A2**, respectively. Kuhn's formulation makes use of ∇, defined by negating Δ (as in note 1):

Ku1 $\quad \Delta\neg p \to \Delta p$

Ku2 $\quad (\Delta p \wedge \nabla(p \vee \underline{\underline{q}})) \to \Delta(\neg p \vee \underline{r})$

Ku3 $\quad (\Delta p \wedge \nabla(p \wedge q)) \to \nabla q$

with, for Δ, the congruentiality rule (**RE**): from $A \leftrightarrow B$ to $\Delta A \leftrightarrow \Delta B$ and the necessitation-like encountered above and called by Kuhn (**R**Δ): from A to ΔA. In the case of Zolin [59] we have Kuhn's (**R**Δ) as the sole Δ-specific rule, and, over the classical non-modal basis, axioms the following (again, not using his own labelling here):

Zo1 $\quad \Delta p \leftrightarrow \Delta\neg p$

Zo2 $\quad \Delta p \to (\Delta(\underline{\underline{q}} \to p) \vee \Delta(p \to \underline{r}))$

Zo3 $\quad \Delta(p \leftrightarrow q) \to (\Delta p \leftrightarrow \Delta q)$

Note that (**RE**) is then derivable with the aid of Modus Ponens, (**R**Δ) and **Zo3**, so any further extension on this basis continues to yield a congruential Δ-logic. In Fan et al. [17], p. 82 (their Def. 4.1), to the same non-modal basis along with the two Δ-rules of Kuhn's axiomatization above, are added the following axioms:

Fa1 $\quad \Delta p \leftrightarrow \Delta\neg p$

Fa2 $\quad \Delta p \to (\Delta(p \to \underline{\underline{q}}) \vee \Delta(\neg p \to \underline{r}))$

Fa3 $\quad (\Delta(p \to q) \wedge \Delta(\neg p \to q)) \to \Delta q$

The first axioms in the **Ku**, **Zo**, and **Fa** axiomatizations (as presented here), say something about how Δ commutes with $\neg$ as do **MRa1**, **MRb1**, and **LG1**, with **Ku1** extracting only one half of the equivalence concerned (as does **MRb1**), in the knowledge that the converse will be forthcoming thanks to congruentiality (or the availability of (**RE**): just substitute $\neg p$ for p). Extracting (for axiomatic purposes) instead, the other direction of the biconditional formulation (as in **MRb1**), however, offers greater clarity in presenting us with the special $\# = \neg$ case of the following principle that could be laid down for all n-ary Boolean primitives #, here expressed with the $\vdash$-notation introduced before Definitions 1.3, and with schematic letters rather than with propositional variables:

(Supervenience) $\qquad \Delta A_1, \ldots, \Delta A_n \vdash \Delta\#(A_1, \ldots, A_n)$.

[54]On this page, all five occurrences of "**K4Δ**" (as well as that on line 4 of p. 233) should be "**KΔ**" (Kuhn's official name for what is called K^{Δ} here), though there is also one reference on p. 231 to the system simply as **K** where again what was intended was **KΔ**.

[55]The same change is made in the case of Fan et al. [17] below, too.

Here we have supervenience in the philosophical sense, as *preservation of agreement*: if all points accessible to a given point (in some model) agree in respect of the truth-values of each of the A_i, they must all agree on the truth-value of $\#(A_1, \ldots, A_n)$.[56] If we reformulate **Ku3** to get rid of the ∇:

$$\textbf{Ku3}' \qquad (\Delta p \wedge \Delta q) \rightarrow \Delta(p \wedge q),$$

we can see it as a representative instance of the above Supervenience schema for the case of $\ = \wedge$. The $n = 0$ form of the schema (in which the A_i disappear and we have only the r.h.s./consequent) gives $\Delta\top$, which, given congruentiality, delivers closure under Kuhn's rule (**R**Δ), and so on. (**Fa3** follows from **Ku3′** by keeping the antecedent of **Fa3** as it is and then putting as its consequent the conjunction of the two Δ-governed formula in the antecedent, noting that this conjunction is provably equivalent to the q in the consequent of **Fa3** itself.)

Ku2, **Zo2** and **Fa2** are the variable-isolating axioms, each isolating two variables. Reformulating **Ku2** to avoid the awkward asymmetrizing effect of the use of ∇ gives us

$$\textbf{Ku2}' \qquad \Delta p \rightarrow (\Delta(p \vee \underline{\underline{q}}) \vee \Delta(\neg p \vee \underline{r})),$$

in which it is easier to see the upshot of this axiom, as well as the fact that **Zo2** and **Fa2** are minor variations on it. **Ku2′** with its form with its disjunctions is better suited to these others, for Kuhn's own definition of the accessibility relation he defines on the canonical models used for the completeness proofs in [32] for K^{Δ} and $\mathsf{K4}^{\Delta}$, however. This uses the idea that one can simulate the notion of necessity sufficiently well for the sake of such proofs by taking the necessity of A (at a point) to consist in the noncontingency of all formulas $A \vee B$, since noncontingency is a matter of necessity or impossibility and whereas for any necessary A all such a $A \vee B$ will again be necessary and hence noncontingent, for an impossible A, disjoining A with some contingent B will give a contingent disjunction. This motivates the definition of the canonical accessibility relation as R^{Ku} in Definition 4.6(i) below – which could certainly be reformulated to a suit the $\rightarrow$-based rather than $\vee$-based **Zo2** and **Fa2** (as indeed is done by Zolin and by Fan et al.). But sticking with the Kuhn version, we can now see what the isolated variables in **Ku2′** are doing and why there are two of them. For this axiom guarantees that when we are at a point (= maximal K^{Δ}-consistent set of formulas) where we have (as an element) ΔA, then for any formulas B, C the point will contain $\Delta(A \vee B) \vee \Delta(\neg A \vee C)$. And this is equivalent to saying that either for every formula B, the point contains $\Delta(A \vee B)$ or for every formula B (as we may now safely put for the previous metalinguistic "C") the formula $\Delta(\neg A \vee B)$, making A *necessary* in the former case (true at every R^{Ku}-related point) or *impossible* in the latter (since $\neg A$ is now true at every such point). We could not have reasoned in this way, thereby completing the 'membership implies truth' argument for the case of Δ-formulas, if we had used q twice over in the axiom, since we don't in that case get the separately $\forall$-quantified disjuncts.

[56] See Fan [12], §§1–2 of Humberstone [30], and other work cited in these sources.

R 4.2 Does the preceding discussion show that, where L is the Δ-congruential logic with axioms **Ku1**, **Ku3**, to extend L axiomatically to K^Δ we need to use an axiom in which two isolated variables appeared? While that is a natural choice in an axiomatization, in view of its role in the completeness proof, all we actually need is that **Ku2′** (or some close variant) should be *provable*: it has not been shown that even shown that $\langle \mathsf{L}, \mathsf{K}^\Delta \rangle$ has the isolation property. For the fact that $\mathsf{L} \oplus$ **Ku2′** $= \mathsf{K}^\Delta$ to play this role we would need to have shown that the isolated variables in question are not removable (Defs. 0.1(*i*) and 2.5), and not only has this not been shown – it is not even true; either of **Ku2′**'s isolated variables is removable in the simplest possible way: by substitution (Def. 0.1(*ii*)). By way of illustration, let us remove r from **Ku2′** by replacing it with q, which has the effect that there are now no isolated variables in the resulting formula, which of course is derivable from **Ku2′** by Uniform Substitution:

$$\textbf{Ku2*} \qquad \Delta p \to (\Delta(p \vee q) \vee \Delta(\neg p \vee q)).$$

As was observed in Humberstone [23], p. 110*f.*, and more recently recalled in Fan [12], note 6, we can recover the original **Ku2′** from this by substitution of $(p \vee q) \wedge (\neg p \to r)$ for q. (In fact the discussion in the cited sources substitutes for q, instead, the equivalence form $(\neg p \to q) \wedge (p \to r)$ for q – or more accurately a version of this with schematic letters rather than propositional variables, and a more closely parallel version of that discussion would have us deriving not **Ku2***, but a formula with a new sentence letter s, say, rather than q, which is perhaps easier to follow, but from which we in turn recover the original **Ku2*** by substituting q uniformly for s. We need to end up with no additional variable-types than occur in the original formula just to abide by the conditions of Def. 0.1(*i*) – since that was the device by which the definition conveniently blocked the proliferation of new variables in the course of removing an existing variable; see Remark 0.2.) All that has been used here are truth-functional manipulations, uniform substitution and congruentiality, so certainly we have $\mathsf{L} \oplus$ **Ku2*** $= \mathsf{K}^\Delta$ for the above L, justifying the claim that r is removable from **Ku2′** for the extension of L to K^Δ. A similar situation applies in the cases of the doubly-isolating axioms **Zo2**, **Fa2**, in their respective axiomatic habitats. ◀

The kind of explanation offered above for the presence of the pairs of isolated variables in the second axioms of the **Ku**, **Zo** and **Fa** axiomatizations – their removability from those axioms (Remark 4.2 notwithstanding) – can perhaps be extended to the case of the *single* isolated variables in **MRb3** and **LG3** and the Section 0 examples of $\Delta 4$, $\Delta 5$, ΔB, via the canonical model considerations (especially when the characterization of accessibility provided by Def. 4.6(*i*) below is used). There is a bit more to say about the general issue of removability (by substitution) of isolated variables too, though, which is most easily said for the cases in which there is only one such variable involved, before we return specifically to the single **MRb3**–**LG3** case; the $\Delta 4$–$\Delta 5$ examples are not addressed here.

We begin with first issue, removal by substitution, seen especially with Boolean constants $\top$, $\bot$, for propositional variables in Sections 1, 2, and most recently with variable-for-variable substitution in Remark 4.2. One might wonder about whether such isolated variables, one per axiom as in the preceding paragraph, rather than with the case of doubly isolating formulae as in Remark 4.2 might be removable by substitution in the presence of *additional* nullary connectives, over and above the familiar Boolean constants. By way of elaboration, recall that, when removal by substitution came up in Definition 0.1(*ii*), the only options on this front were the constants $\top$ and $\bot$ (and Boolean combinations of their modalizations, which there brought nothing new). On the other hand, Claudio Pizzi made the striking observation that the addition of a new constant τ, with, in the simplest case, the axiom $\nabla\tau$, allowed us to recover the 'underlying' $\Box$ by putting $\Box A = \Delta A \wedge \Delta(\tau \to A)$, which could equally well have $\vee$ rather than $\to$ on the r.h.s.[57] Less simply, but with a view to having K rather than KD emerge from the extension of K^{Δ} as the induced monomodal $\Box$-based logic, Pizzi offers a weakening of the τ axiom $\nabla p \to \nabla\tau$, which, interestingly enough, is again a variable-isolating formula. However, the point of recalling Pizzi's treatment(s) here was not to suggest that it was τ itself that might serve to remove an isolated variable, but simply as a reminder that (even one-dimensional) modal logic offers considerable room for the addition of sentential constants that would be worth exploring with that end in mind.[58]

Turning now to the matter of the single isolated variable **MRb3** or its variant **LG3**, the role of that isolated variables is more easily explained by looking at yet another variant, one mentioned already in this capacity in note 53, namely that mentioned in the following:

P 4.3 $\mathsf{KT}^{\Delta} = \mathsf{K}^{\Delta} \oplus (\Delta p \wedge p) \to \Delta(p \vee q)$.

Proof. The (q-isolating) formula on the right is easily seen to be valid on every reflexive frame, giving the $\supseteq$-direction. But for a more hands-on approach, let us exhibit a deduction of $(\Delta p \wedge p) \to \Delta(p \vee q)$ from (K^{Δ} and) $\Delta\mathsf{T}$. We substitute $p \vee q$ for q in the latter, getting

$$p \to (\Delta(p \to (p \vee q)) \to (\Delta p \to \Delta(p \vee q))).$$

Since the second antecedent, $\Delta(p \to (p \vee q))$, is K^{Δ}-provable, we can delete it, to obtain a truth-functional reformulation of the desired conclusion.

[57] See Pizzi [43], and for some discussion, esp. §4 of Humberstone [28], to which I would add the following remark. Pizzi's way of validating to the $\nabla\tau$ axiom is to have the frames distinguish a non-empty proper subset of their universes, to serve as τ truth-set in every model, and require each point to have accessible to it some element of this set and some element of its complement (relative to the frame's universe). But rather than having some one fixed contingent proposition (this distinguished set of points) it would suffice for validating the axiom to have, for every point, a proposition contingent at that point, and interpret τ at $x \in W$ as picking out that proposition. This amounts to taking the sentential constant τ two-dimensionally, not as a propositional constant but as standing for what had variously been called – or better, variously conceived as – a di-proposition or a propositional concept (explanations of which terms together with appropriate references can be found on p. 53, including note 27, of [24]). This does, however, potentially create obstacles to closure under uniform substitution unless a similar generalization is made for the sentence letters, now less suitably called propositional variables. But it does give greater flexibility. For instance, in a three-element frame in which the accessibility relation coincides with non-identity, at each point there is a contingent proposition (take the unit set of one of the other points), while there is no proposition contingent at each point, even though we can still find for each point, a proposition contingent at that point.

[58] Recent attention has been paid to the role of such constants in modal logic in French [18] and Goldblatt and Kowalski [21]; for a further discussion of specifically two-dimensionally treated sentential constants/nullary connectives, see Humberstone [29], p. 275*ff*.

Conversely, we derive $\Delta\mathsf{T}$ from (K^{Δ} and) the new formula, $(\Delta p \wedge p) \to \Delta(p \vee q)$, taking $\Delta\mathsf{T}$ in the following variant form, so as to have an antecedent matching that of the new formula:

$$(\Delta p \wedge p) \to (\Delta(p \to q) \to \Delta q).$$

We sketch a formal proof by giving an informal natural deduction argument (as in the proof of Lemma 3.1), which begins by assuming (a) the antecedent of the formula just inset, with a view to deriving its consequent, which is itself conditional in form so we assume, in turn, *its* antecedent, (b) ($=\ \Delta(p \to q)$). From (a) and the new formula, we infer $\Delta(p \vee q)$, and we can rewrite (b) as $\Delta(\neg p \vee q)$; so from these two, by (Supervenience), or more specifically, by **Ku3′** (and Uniform Substitution) we derive

$$\Delta((p \vee q) \wedge (\neg p \vee q)),$$

and thus, by congruentiality Δq, so we can discharge assumption (b) and conclude that we have derived $\Delta(p \to q) \to \Delta q$ from assumption (a), and thus, discharging this assumption, that the variant form is provable in $\mathsf{K}^{\Delta} \oplus (\Delta p \wedge p) \to \Delta(p \vee q)$. ■

R 4.4 The second half of the proof of Proposition 4.3 makes use of **Ku3′** to reflect the fundamental status of (Supervenience), of which **Ku3′** is a special case. A shorter proof is available (as Fan has pointed out in correspondence) if we make use of the **Fa** axiomatization and, in particular, of **Fa3** at this point, because the following variant of the formula mentioned in Prop. 4.3:

$$(\Delta p \wedge p) \to \Delta(\neg p \to q),$$

taken together with **Fa2**, has $\Delta\mathsf{T}$ as a truth-functional consequence. ◀

The isolated q in the new reflexivity axiom figuring in Proposition 4.3 makes perfect sense from the perspective of our discussion of **Ku2′** above, in which the ('Δ-surrogate') necessity of A is cashed out as the totality of disjunctions, in the scope of Δ of A with each formula B, so using

$$(\Delta p \wedge p) \to \Delta(p \vee \underline{\underline{q}}),$$

as an axiom allows us for a given A, to conclude from the presence of $\Delta A \wedge A$ in any deductively closed (e.g., any maximal consistent) set, that $A \vee B_1, \ldots, A \vee B_n, \ldots$ are also present, where the B_i exhaust the set of formulas. So this means that the – in any reflexive model – genuine necessity of A expressed by $\Delta A \wedge A$ implies its noncontingency surrogate and indeed that in the canonical model, the truth of A in all accessible points (something best seen using accessibility as defined in 4.6(*i*) below).[59]

[59]Further discussion along these lines, however, would have to come to terms with the awkward fact that the frame of the canonical model, as currently conceived, is not actually reflexive, by contrast with the situation $\Box$-based modal logic – as was pointed out in §3 of Zolin [59]. (The "as currently conceived" is included here because one might work instead with a different canonical accessibility relation. This is done in Fan [14] for extensions of KT^{Δ}, where also one can find a reflexivity-oriented version, $\Delta\mathsf{B}'$ of the symmetry axiom $\Delta\mathsf{B}$ given in Section 0 above.)

So much for the explanatory remarks on the isolating KT^{Δ} axioms. A further comment on **Ku2′** is in order before we return to the topic of accessibility in the canonical models. Using the equivalence of ΔA with $\Delta\neg A$ on the second disjunct of the consequent of **Ku2′**, we see that we could equally well write this formula as

$$\textbf{Ku2}'' \qquad \Delta p \to (\Delta(p \vee q) \vee \Delta(p \wedge \neg r))$$

or indeed, without the negation on r, and in the schematic style (Supervenience) above:

(Co-convexity) $\quad \Delta A \vdash \Delta(A \vee B) \vee \Delta(A \wedge C)$.

The label used here can be taken to apply to an arbitrary condition on 1-ary connectives O in the language of a consequence relation $\vdash$ (appearing where Δ appears here), and is intended to recall the dual condition on such an O:

(Convexity) $\quad O(A \vee B) \wedge O(A \wedge C) \vdash OA$.

The latter is equivalent, in a congruential setting, to the condition that (for all D_1, D_2, D_3) if $D_1 \vdash D_2$ and $D_2 \vdash D_3$, then $OD_1, OD_3 \vdash OD_2$: i.e., that any $\vdash$-closed set of formulas is convex in the sense of being closed under 'inferential betweenness'.[60] One checks easily that any 1-ary operator O satisfies this second condition whenever OA can be written as the conjunction of a monotone O_{m} with an antitone O_{a}, as is the case of ∇ in K with $O = \nabla$ taken as so defined ($O_{\mathsf{m}} = \Diamond$, $O_{\mathsf{a}} = \Diamond\neg$). Similarly with co-convexity and disjunction, as arises for $O = \Delta$ ($O_{\mathsf{m}} = \Box$, $O_{\mathsf{a}} = \Box\neg$). It is also worth noticing that (Co-convexity) is satisfied when Δ is read not as the disjunction of $\Box$ and $\Box\neg$ but as their hybrid (*à la* note 27), meaning that both necessity and impossibility, as they behave in K, for definiteness (more generally: any congruential extension of EM), satisfy this condition. (For this hybrid logic, though, we lose the connection with **Ku2′** since ΔA and $\Delta\neg A$ are no longer equivalent.[61] As well as being convex, each of $\Box$ and $\Box\neg$ – and thus their hybrid – is also co-convex, though in this case Δ does not follow suit – put $A = p$, $B = \top$, $C = \bot$ in (Convexity), with O as Δ.) Thus Kuhn's axiomatization **Ku1**–**Ku3**, slightly tweaked, can be seen as showing that the basic noncontingency logic, K^{Δ}, is the smallest congruential modal logic satisfying, for its non-Boolean primitive Δ, the conditions (Supervenience) and (Co-convexity).

To conclude this discussion, let us look at the way the canonical accessibility relations for K^{Δ} and its (congruential) extensions are defined by Kuhn and by Fan et al., since, as remarked above, these do seem to throw promising light on the distinctive appearance of isolated variables in the more economical axiomatizations of these logics – whatever the eventual fate of the isolation property/strong isolation property distinction may be. An earlier paper by the present author contained a more complicated definition of the accessibility relation for K^{Δ}'s canonical model than is to be found in Kuhn [32], but the question of whether Kuhn's simplification pertained to the relation

[60] These conditions are from p. 584 of Humberstone [27]; in fact, Co-convexity axioms and rules appeared in [23], p. 111, in essentially this connection, though without that (or any other) label.

[61] This shows that the present hybrid – or intersection – logic is not included in K^{Δ}. To see that the converse inclusion also fails, consider $(\Delta p \wedge \Delta\neg p) \to \Delta q$. In fact, as one gets with the intersection of any two $\subseteq$-incomparable logics, the hybrid Δ satisfies not only a representative instance of the Co-convexity schema $\Delta p \to (\Delta(p \vee q) \vee \Delta(p \wedge q))$ but the stronger 'Halldén unreasonable' version: $(\Delta p \to \Delta(p \vee q)) \vee (\Delta r \to (\Delta(r \wedge s)))$.

defined or just to the *way* it was defined was only explicitly addressed more recently, in §5 of Fan [13], where it is shown that the latter is the case: whichever definition is employed, one gets the same relation between maximal consistent sets of formulas relative to K^{Δ} (and its congruential extensions).[62] The same question arises over the relationship between canonical accessibility as defined by Kuhn and as defined by Fan and co-authors (see Defs. 4.6(*i*) and (*iii*)) – for example it looks on the face of it that Fan et al.'s accessibility relation differs from Kuhn's in respect of guaranteeing that any point with a successor (by the relation) has at least two successors. As with the earlier double definition case, though, it is not explicitly stated whether we are faced with two relations or just two ways of defining the same relation. We close by showing that the latter is the case, beginning with an observation for later use, which does not require the concepts introduced in Definitions 4.6(*i*) and (*iii*):

L 4.5 $\Delta A, \Delta(D \to A) \vdash_{\mathsf{K}^{\Delta}} \Delta D \vee \Delta(C \to A)$, *for all formulas* A, C, D.

Proof. This can be easily checked for validity on an arbitrarily selected frame. ■

D 4.6 Variables x, y range over arbitrary sets of formulas of the Δ-based language (though for our applications below, these will be sets which are maximal consistent w.r.t to the logic K^{Δ}), and A, B, C over arbitrary formulas of that language. We use & and ⇒, for conjunction and implication in the metalanguage (with ∀ and ∃ as quantifiers):

(*i*) $R^{\mathsf{Ku}}xy$ iff $\forall A[\forall B\, \Delta(A \vee B) \in x \;\Rightarrow\; A \in y]$.

(*ii*) $R^{\mathsf{Fa}}_{C}xy$ iff $\forall A[\Delta A \wedge \Delta(C \to A) \in x \;\Rightarrow\; A \in y]$.

(*iii*) $R^{\mathsf{Fa}}xy$ iff $\exists C[\nabla C \in x \;\&\; R^{\mathsf{Fa}}_{C}xy]$. ◀

The superscripted letters in the definienda for (*i*) and (*iii*) here are meant to suggest that we are dealing with the canonical accessibility relation as defined in Kuhn [32] and by Fan et al. (e.g. [16], [10]), with a convenient relation exhibiting a witness for the existential quantification in (*iii*) isolated in (*ii*).[63] Lemma 4.7 and Proposition 4.8 provide

[62] [13] shows this in an even more general setting, with a weaker background logic, called there and in Fan and van Ditmarsch [15] E^{Δ}, and encountered above in passing in Remark 4.1. This is the Δ-fragment of the smallest congruential $\Box$-based modal logic E, with ΔA primitive but behaving as though defined in E by $\Box A \vee \Box\neg A$, rather than being the smallest Δ-based modal logic in which Δ is congruential. An axiomatic description is provided on p. 96 of [15], using the Δ-congruentiality rule and the axiom schema $\Delta A \leftrightarrow \Delta\neg A$. In fact even the latter schema is not needed – just congruentiality: see [13], p. 695. The following discussion of accessibility in the canonical models addresses only the case of K^{Δ} and its congruential extensions. (Thanks to Jie Fan for drawing my attention to these last two points.) Given congruentiality, what is exploited in the proof in [13] is the fact that the inference from a disjunct to a disjunction is classically archetypal. (See note 24 and references given there.) Since the same goes for the inference from a conjunction to one of its conjuncts, one expects to see a kind of dual variant of Def. 4.6(*i*) below using conjunction instead of disjunction. And indeed, for a formula A, "$\forall B\, \Delta(A \wedge B) \in x$" amounts to saying that A is impossible at x, so the envisaged variant would say that for every A for which this is the case, we have $A \notin y$. So to simulate the necessity of A along these lines we would use: for all B, $\Delta(\neg A \wedge B)$. This looks rather different from saying: for all B, $\Delta(A \vee B)$, but the $\Delta/\Delta\neg$ equivalence and some De Morgan reveal them to be equivalent for any given A. This time, though, we are using the full force of E^{Δ} and not just the congruentiality of Δ.

[63] Note 5 of Fan et al. [16] remarks on the connection between R^{Fa}, as defined in (*iii*) and the use of Pizzi's constant, from note 57 above and the text to which it is appended.

further information on the characterization supplied in (*i*) and (*ii*). In the former case, it is just a matter of spelling out the well known equivalence of two characterizations given of the accessibility relation in the canonical models of consistent $\Box$- (or $\Diamond$-)based normal modal logics: (a) Rxy iff for all A, $\Box A \in x \Rightarrow A \in y$ and (b) Rxy iff for all A, if $A \in y \Rightarrow \Diamond A \in x$.

L 4.7 *The following conditions define the same binary relation between maximal consistent sets of formulas relative to any consistent congruential extension of* K^{Δ}*:*

(a) $\forall A[\forall B\Delta(A \vee B) \in x \Rightarrow A \in y]$ (b) $\forall A[A \in y \Rightarrow \exists B\nabla(A \to B) \in x]$.

Proof. (a) evidently implies that for all A, if $\forall B\Delta(\neg A \vee B) \in x \Rightarrow \neg A \in y$, contraposing which and using the maximal consistency of the sets involved, we have for all A, $A \in y$ implies that not $\forall B\Delta(\neg A \vee B) \in x$, and hence that $\exists B\Delta(\neg A \vee B) \notin x$, which we can reformulate to: $\exists B\nabla(A \to B) \in x$, as in (b). That (b) $\Rightarrow$ (a) is established similarly. ■

Thus instead of defining $R^{\mathsf{Ku}}xy$ as in Def. 4.6(*i*), by means of (a) in Lemma 4.7, we could equally well have chosen (b). The following observation helps to clarify the question of how sensitive Definition 4.6 is to the choice of the formula in the subscript position.

P 4.8 *For all* K^{Δ}*-maximal consistent sets* x, y *and all formulas* C, D *such that* $\nabla C, \nabla D \in x$*, we have* $R^{\mathsf{Fa}}_{C}xy$ *if and only if* $R^{\mathsf{Fa}}_{D}xy$*.*

Proof. Suppose we have x, y as described, with $\nabla C, \nabla D \in x$ and $R^{\mathsf{Fa}}_{C}xy$, with a view to showing that $R^{\mathsf{Fa}}_{D}xy$, the converse implication being secured by the symmetry of the formulation. (For this first direction, we actually only need $\nabla D \in x$, with $\nabla C \in x$ for the converse.) Thus, we are supposing

$$\forall A[\Delta A \wedge \Delta(C \to A) \in x \Rightarrow A \in y] \tag{*}$$

and want to show that for an arbitrarily selected formula A, (1) $\Delta A \wedge \Delta(D \to A) \in x$ implies (2) $A \in y$. The conjuncts of (1) can be taken as those represented by the same letters in Lemma 4.5, so we can conclude in view of that Lemma that $\Delta D \vee \Delta(C \to A) \in x$. But since the first disjunct here does not belong to x since $\nabla D \in x$ (i.e., $\neg\Delta D \in x$), it's the second disjunct, $\Delta(C \to A)$, that belongs to x, as does ΔA, the first conjunct of (1). Thus we have $\Delta A \wedge \Delta(C \to A) \in x$ and so by our supposition (*), we have the desired conclusion, (2). ■

T 4.9 *As relations between sets of formulas maximal consistent w.r.t. any congruential extension of* K^{Δ}, $R^{\mathsf{Ku}} = R^{\mathsf{Fa}}$.

Proof. $R^{\mathsf{Fa}} \subseteq R^{\mathsf{Ku}}$: Suppose that for x, y we have $R^{\mathsf{Fa}}xy$ in order to show that $R^{\mathsf{Ku}}xy$. Since $R^{\mathsf{Fa}}xy$, by Def. 4.6(*ii*), (*iii*) there is some C for which $\nabla C \in x$ and for all formulas A, we have (*) from the proof of Prop. 4.8. From these assumptions we want to conclude that $R^{\mathsf{Ku}}xy$, i.e., that supposing for an arbitrarily chosen A, if (1′) for all B,

$\Delta(A \vee B) \in x$, then (2′) $A \in y$. Selecting B first as A itself (or indeed as $\bot$) and then as $\neg C$, from (1)′ we have $\Delta A \in x$ and $\Delta(C \to A) \in x$ (appealing to congruentiality in each case), so (*) gives us the desired conclusion, (2)′.

Next $R^{\mathsf{Ku}} \subseteq R^{\mathsf{Fa}}$: Suppose that for x, y we have $R^{\mathsf{Ku}}xy$, with a view to showing that $R^{\mathsf{Fa}}xy$. By Lemma 4.7, $R^{\mathsf{Ku}}xy$ implies that for all $A \in y$ we can find a formula B with $\nabla(A \to B) \in x$, though to avoid notational confusions, we will rephrase this as saying that for each $A \in y$ we have a formula C with $\nabla(A \to C) \in x$, since such a C will be seen to behave as required for the C of Definition 4.6(*iii*). As a first step toward seeing this, choose $A = \top$, since certainly $A \in y$ for this choice. Then, replacing $\top \to C$ with the equivalent C, we see that we have $\nabla C \in x$. Now, forgetting the A just chosen, we suppose now that with this C fixed, we have, for some formula A the antecedent of (*) from the proof of Prop. 4.8: $\Delta A \wedge \Delta(C \to A) \in x$, with a view to showing that $A \in y$, and thus that $R^{\mathsf{Fa}}xy$. Much as in that proof, we now appeal to Lemma 4.5, omitting the quantifiers and choosing the letters so that "C" appears appropriately for current purposes:

$$\Delta A, \Delta(C \to A) \vdash_{\mathsf{K}^{\Delta}} \Delta C \vee \Delta(E \to A),$$

and, putting $\neg B$ for E so that the rightmost Δ-formula can be taken as $\Delta(B \vee A)$, whose disjuncts we now exchange to match the order in Def. 4.6(*i*):

$$\Delta A, \Delta(C \to A) \vdash_{\mathsf{K}^{\Delta}} \Delta C \vee \Delta(A \vee B).$$

By our supposition the formulas on the left belong to x and so therefore does the disjunction on the right. But since $\nabla C \in x$, the first disjunct does not belong to x, so for all B, $\Delta(A \vee B) \in x$, and thus, since A was arbitrary and $R^{\mathsf{Ku}}xy$, we have $A \in y$. ■

We close with reminders of some of the main open problems our discussion has left us with. Note 17 mentions the question of (what we are now calling) the isolation property for varying choices of primitives, for the extension of intuitionistic logic to this or that intermediate logic, for which case closure under Modus Ponens should play the role played by the closure conditions on congruential modal logics which is packed into the "$\oplus$" notation has the isolation property.[64] Then there is the matter of whether the extension pair $\langle \mathsf{E}, \mathsf{EM} \rangle$ has the isolation property, note 24 in effect conjecturing that this is so. There is also, while not exactly an open problem in the technical sense, the invitation to explore the possibility of using novel sentential constants to enhance the procedure of removal of isolated variables by substitution, raised in the text to which note 58 is appended. An especially pressing issue, perhaps, is raised by the Open Question in Section 3, after Proposition 3.2, or the variant mentioned below the formulation of that question, with other primitive Boolean connectives: give an

[64] If IL_+ and CL_+ are respectively the $\{\to \wedge, \vee, \top\}$-fragments of intuitionistic and classical propositional logic, then clearly $\langle \mathsf{IL}_+, \mathsf{CL}_+ \rangle$ has the isolation property because we can effect the extension in question using Peirce's law, from which the isolated variable is not removable since these logics are alike in respect of their (at most) one-variable theorems. But in the classical case our main discussion has assumed a functionally complete set of Boolean primitives, so the corresponding expressive completeness condition should be in force here too: any change of primitives should give a logic definitionally equivalent to full propositional IL in the more familiar primitives.

example of such a connective, or show that none can be given, whose presence makes available an extension $\langle \mathsf{EM}, \mathsf{EM} \oplus A(\dot{p})\rangle$ with the isolation property. We also have the KT^{Δ} Conjecture mentioned in Remark 4.1 as well as numerous issues raised at the end of that Remark, and the overarching problem behind it (from the discussion after Proposition 2.11): to give an example of an extension pair with the isolation property but without the strong isolation property. Finally, there were also minor conjectures in the paragraph after the **LG** axioms in this section.

5 Appendix: Proof of Lemma 2.1

Here we repeat the content of Lemma 2.1 before sketching its (routine) proof, aspects of which are deferred to Remarks 5.1(*i*) and (*ii*) below, to avoid clutter:

L 2.1 *Let* L *be any monotonic modal logic, with Boolean primitives as in Examples 1.5. Then*

(1) *If all occurrences of* p_i *in a formula* $A(p_i)$ *are positive, then* $A(p_i)$ *is monotone according to* L, *i.e., for all formulas* B, C, *if* $B \vdash_{\mathsf{L}} C$ *then* $A(B) \vdash_{\mathsf{L}} A(C)$*;*

(2) *If all occurrences of* p_i *in* $A(p_i)$ *are negative, then* $A(p_i)$ *is antitone according to* L, *i.e., for all formulas* B, C, *if* $B \vdash_{\mathsf{L}} C$ *then* $A(C) \vdash_{\mathsf{L}} A(B)$*.*

Proof. Basis case: the complexity of $A(p_i)$ is 0. This means there are no connectives in $A(p_i)$, so, setting aside a degenerate possibility we can consider later (Remark 5.1(*i*)), with p_i not occurring in $A(p_i)$ at all, $A(p_i)$ is just the formula p_i, in which p_i certainly occurs positively and moreover (1) is satisfied, because if $B \vdash_{\mathsf{L}} C$ then $A(B) \vdash_{\mathsf{L}} A(C)$ since $A(B)$ and $A(C)$ are just B and C respectively. (2) is also satisfied because the antecedent "all occurrences of p_i in $A(p_i)$ are negative" is false, the one occurrence of p_i in the current $A(p_i)$ is not negative.

Inductive step: Suppose that the result holds for all $A(p_i)$ of lower complexity than n, where $n > 0$, with a view to showing that it must hold for $A(p_i)$ of complexity n. Taking our primitives to be $\Box$ and the Boolean connectives mentioned in the statement of the Lemma, we have, setting aside the nullary connectives $\top$ and $\bot$ which will be addressed after the main proof, connectives $\wedge, \vee, \rightarrow$, and $\neg$ to deal with alongside $\Box$. For illustrative purposes (though there is nothing new here) we will explicitly address the cases of $\neg$, $\wedge$ and $\Box$.

($\neg$) $A(p_i)$ is $\neg A_0(p_i)$, where of course $A_0(p_i)$ is of lower complexity than $A(p_i)$, so we are entitled to assume that (1) and (2) of the Lemma hold for $A_0(p_i)$. To show (1) for $A(p_i)$, assume that all occurrences of p_i in $A(p_i)$ are positive, with a view to showing that $A(p_i)$ is monotone (in L), i.e. if (*) $B \vdash_{\mathsf{L}} C$, we can conclude that (**) $A(B) \vdash_{\mathsf{L}} A(C)$. Since all occurrences of p_i in $A(p_i)$ ($= \neg A_0(p_i)$) are positive, all such occurrences in $A_0(p_i)$ are negative (Lemma 3.37), so the inductive hypothesis – appealing to (2) for $A_0(p_i)$ – gives us that $A_0(p_i)$ is antitone, so, from (*) it follows that $A_0(C) \vdash_{\mathsf{L}} A_0(B)$, and from this by non-modal classical propositional logic we infer ('contraposition') that $\neg A(B) \vdash_{\mathsf{L}} \neg A_0(C)$. But this is the desired conclusion (**).

Next, to show (2) for $A(p_i)$, assume that all occurrences of p_i in $A(p_i)$ are negative, with a view to showing that $A(p_i)$ is antitone (in L), i.e. if (†) $B \vdash_{\mathsf{L}} C$, we can conclude that (††) $A(C) \vdash_{\mathsf{L}} A(B)$. Now all occurrences of p_i in $A_0(p_i)$ are positive, so by the inductive hypothesis, (†) gives us that $A_0(B) \vdash_{\mathsf{L}} A_0(C)$, as before we contrapose to conclude that $\neg A_0(C) \vdash_{\mathsf{L}} \neg A_0(B)$, which is (††).

($\wedge$) The second inductive case to consider is that of $A(p_i) = A_0(pi) \wedge A_1(p_i)$, where again we are entitled to assume that (i) and (ii) of the Lemma hold for each of the conjuncts. To show (1) for $A(p_i)$, assume that all occurrences of p_i in $A(p_i)$ are positive, with a view to showing that $A(p_i)$ is monotone (in L), i.e. if (*) $B \vdash_{\mathsf{L}} C$, then we can conclude that (**) $A(B) \vdash_{\mathsf{L}} A(C)$. Since all occurrences of p_i in $A(p_i)$ are positive, all occurrences of p_i in $A_0(p_i)$ are positive, and so are all such occurrences in $A_1(p_i)$, so by the inductive hypothesis (specifically, (1) for $A_0(p_i)$ and for $A_1(p_i)$) we have $A_0(B) \vdash_{\mathsf{L}} A(C)$ and $A_1(B) \vdash_{\mathsf{L}} A(C)$, from which it evidently follows that $A_0(B) \wedge A_1(B) \vdash_{\mathsf{L}} A_0(C) \wedge A_1(C)$. But this is (**).

Next, for (2) for $A(p_i)$, suppose that all occurrences of p_i in $A(p_i)$ are negative, in order to show that (†)$B \vdash_{\mathsf{L}} C$, implies (††) $A(C) \vdash_{\mathsf{L}} A(B)$. Since p_i occurs only negatively in $A(p_i)$, all occurrences of p_i in $A_0(p_i)$ and all occurrences of p_i in $A_1(p_i)$ are negative, so by the inductive hypothesis, (†) implies $A_0(C) \vdash_{\mathsf{L}} A_0(B)$ and also $A_1(C) \vdash_{\mathsf{L}} A_1(B)$, which together imply $A_0(C) \wedge A_1(C) \vdash_{\mathsf{L}} A_0(B) \wedge A_1(B)$, i.e., (††).

($\Box$) Finally, we have the case of $A(p_i) = \Box A_0(p_i)$. For (1) suppose that all occurrences of p_i in $A(p_i)$ are positive, and that (*) $B \vdash_{\mathsf{L}} C$. with a view to showing (**) $A(B) \vdash_{\mathsf{L}} A(C)$. As all occurrences of p_i in $A_0(p_i)$ must also be positive, by ind. hyp. from (*) we have $A_0(B) \vdash_{\mathsf{L}} A_0(C)$, so since L is monotonic, $\Box A_0(B) \vdash_{\mathsf{L}} \Box A_0(C)$, which is the desired conclusion (**). To show (2) for the present case, suppose that all occurrences of p_i in $A(p_i)$ are negative, and that (†) $B \vdash_{\mathsf{L}} C$. with a view to showing (††) $A(C) \vdash_{\mathsf{L}} A(B)$. Since all occurrences of p_i in $A(p_i)$ are negative, all those in $A_0(p_i)$ are also negative, and by ind. hyp. (†) $B \vdash_{\mathsf{L}} C$ implies $A_0(C) \vdash_{\mathsf{L}} A_0(B)$, whence, by the monotone rule for $\Box$, again, we conclude that $\Box A_0(C) \vdash_{\mathsf{L}} \Box A_0(B)$, which is (††). ■

Now to attend to the complication set aside under the Basis Case above, namely the 'degenerate' possibility (. . .) with p_i not occurring in $A(p_i)$ at all. For a smooth treatment, we need to insist that the notation "$A(p_i)$"-style notation does not require that p_i should actually occur in A, since when dealing with $A(p_i)$ as, for example, a conjunction in which all occurrences of p_i are positive (say) we want to write this as $A_0(p_i) \wedge A_1(p_i)$, and even if p_i occurs positively and only positively in the conjunction, it need not occur in both conjuncts, so the notation has to remain neutral as to the possibility that there is no occurrence of p_i in, say, $A_0(p_i)$. [65]

R 5.1 (*i*) But then the inductive hypothesis needs to be able to handle this case too, and so we must address it at the 'basis' stage. Let us, then, attend to this. Suppose $A(p_i)$ is a formula of complexity 0 in which p_i does not occur. This means that $A(p_i)$ is p_j for some $j \neq i$. Vacuously, all occurrences of p_i in this formula are positive, so we

[65] In fact, overlooking this consideration can lead to confusion, as is illustrated, in connection with the logic of definite descriptions, in French and Humberstone [19], §§1 and 2.

need to show (the monotone property) that (*) $B \vdash_{\mathsf{L}} C$ implies (**) $A(B) \vdash_{\mathsf{L}} A(C)$. Here, too, $A(B)$ is the result of replacing all occurrences of p_i in $A(p_i)$ by B, and similarly, *mutatis mutandis*, for $A(C)$. Thus when there are no occurrences to replace, each of $A(B), A(C)$ is just the original formula $A(p_i)$, which in the present instances is the formula p_j itself, so we have (**) automatically, as this simply says that $p_j \vdash_{\mathsf{L}} p_j$. This also creates a degenerate basis case for (2) of the Lemma, the antitone part, since again vacuously, all occurrences of p_i in $A(p_i)$, when there are no such occurrences, are negative. So we also need to show that when (†) $B \vdash_{\mathsf{L}} C$, we have (††) $A(C) \vdash_{\mathsf{L}} A(B)$, and again without even attending to the supposition (†), we have (††) outright, as this is exactly as before, $p_j \vdash_{\mathsf{L}} p_j$. (Here we use the same asterisk/obelisk – or star/dagger, if you prefer – conventions as in the proof.)

(*ii*) Finally, let us note that since, as with Blackburn et al., we take $\bot$ as one of our primitives (and in our case $\top$ also, to which the present remark applies equally), another minor tweak in the inductive step is called for. Although $\bot$ is an atomic formula – a formula not constructed out of other formulas – it is not a formula of complexity 0, but of complexity 1 (since it contains one connective, namely the nullary connective $\bot$ itself), so we need to check that if $A(p_i)$ is the formula $\bot$ we again have both $A(B) \vdash_{\mathsf{L}} A(C)$ and $A(C) \vdash_{\mathsf{L}} A(B)$ and we do have these for the reason given in the 'complexity 0' case just reviewed: each of $A(B), A(C)$ is just the formula $\bot$ again. Alternatively, these cases can be subsumed under the basis case if we do the induction on (complexity construed as) the number of non-nullary connectives used in the construction of a formula, instead of the number of connectives *tout court* used in its construction. ◀

Acknowledgments. I am greatly indebted to Shawn Standefer, to Rohan French, and to a reader for *Filosofiska Notiser*, and especially to Jie Fan, for their many helpful questions, suggestions and corrections.

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