True Belief and Mere Belief About a Proposition and the Classification of Epistemic-Doxastic Situations

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Abstract

Starting from standard logics of knowledge and belief with principles such as introspection of beliefs and 'knowledge implies belief', we study two non-normal modalities of belief: true belief about a proposition and what we call mere belief about a proposition. We show that these modalities suffice to define all possible epistemic-doxastic situations in a combinatorial manner. Furthermore, we show that two consecutive modalities that are indexed by the same agent can be reduced for two of the three logics of knowledge and belief that we consider.

1 Introduction

Standard logics of epistemic attitudes allow us to reason about propositions of the form "agent i knows that φ is the case" and "agent i believes that φ is the case", formally written $\mathsf{K}_i\varphi$ and $\mathsf{B}_i\varphi$. There is a recent trend to go beyond these two modalities and investigate the logic of other epistemic attitudes such as 'knowing how' and 'knowing who' [32, 33]. We here focus on modalities related to 'knowing whether' and its belief counterpart. Such modalities have been studied since long [26, 20] and have recently gained some interest [13, 25, 18, 19].

In natural language one cannot say "I believe whether φ"; we are therefore going to talk about propositions of the form "i has knowledge about φ" and "i has a belief about φ". Formally, we write $\mathsf{KA}_i\varphi$ for the former and $\mathsf{BA}_i\varphi$ for the latter. We further analyse this and distinguish "i has a true belief about φ" and "i has a mere belief about φ", respectively written $\mathsf{TBA}_i\varphi$ and $\mathsf{MBA}_i\varphi$. We understand $\mathsf{MBA}_i\varphi$ as "i has a belief about φ but does not know whether φ".

The modalities TBA_i and MBA_i are non-standard, but nevertheless natural. To witness, consider the evolution of knowledge and belief in a variant of the famous Sally-Ann Test [34, 4]. Suppose Sally and Ann are in a room. Ann is holding a marble. There is a basket and a box in the room. Let S stand for Sally, A for Ann, and b for "the marble is in the basket". We describe Ann's and Sally's epistemic situations after the following events took place.

1. Ann puts the marble in the basket. Sally then knows that the marble is in the basket, i.e., she has a belief about *b* that is both true and not a mere belief:

$$b \land \mathsf{TBA}_A b \land \neg\mathsf{MBA}_A b \land \mathsf{TBA}_S b \land \neg\mathsf{MBA}_S b.$$

2. Sally leaves the room. Sally continues to believe that the marble is in the basket, but she no longer knows that:

$$b \land \mathsf{TBA}_A b \land \neg\mathsf{MBA}_A b \land \mathsf{TBA}_S b \land \mathsf{MBA}_S b.$$

3. Ann transfers the marble to the box. Supposing that Sally still believes that the marble is in the basket she now has a false belief:

$$\neg b \land \mathsf{TBA}_A b \land \neg\mathsf{MBA}_A b \land \neg\mathsf{TBA}_S b \land \mathsf{MBA}_S b.$$

4. Sally re-enters and looks inside the basket. Now she knows that the marble is not in the basket:

$$\neg b \land \mathsf{TBA}_A b \land \neg\mathsf{MBA}_A b \land \mathsf{TBA}_S b \land \neg\mathsf{MBA}_S b.$$

We could extend the story in such a way that Sally goes through all possible epistemic situations; for example, Sally could initially be outside the room, not having any idea whether or not the marble is in the basket: then both $\mathsf{TBA}_S b$ and $\mathsf{MBA}_S b$ would be false.

Beyond the study of epistemic situations we show in the present paper that the modalities TBA_i and MBA_i are fully expressive, in the sense that both $\mathsf{K}_i\varphi$ and $\mathsf{B}_i\varphi$ can be defined as abbreviations from them. Furthermore, we show that the eight possible epistemic situations w.r.t. a contingent proposition φ can be characterised in terms of the eight possible combinations of the three formulas φ, $\mathsf{TBA}_i\varphi$, and $\mathsf{MBA}_i\varphi$. Finally, we study reductions of consecutive modalities with the same agent. We show that for all three logics, $\mathsf{MBA}_i\mathsf{TBA}_i\varphi$ is equivalent to $\mathsf{MBA}_i\varphi$ and $\mathsf{TBA}_i\mathsf{MBA}_i\varphi$ is equivalent to $\mathsf{TBA}_i\varphi \lor \neg\mathsf{MBA}_i\varphi$. Moreover, we show that for the two strongest of our three logics all four possible combinations can be reduced to modal depth one.

The formal background of our investigation are three epistemic-doxastic logics having both 'knowledge that' and 'belief that' modal operators (Section 2). From these we define several 'knowledge about' and 'belief about' modal operators and show that they have the same expressivity (Section 3). We then give the reductions of consecutive modalities featuring the same agent (Section 4) and conclude (Section 5).

2 Three Epistemic-Doxastic Logics

In this section we recall three epistemic-doxastic logics of 'knowledge that' and 'belief that': one basic system and two possible extensions.

Since the seminal work of Fagin, Halpern and colleagues [16, 12], it is fairly common in artificial intelligence to consider that the modal logic S5 is *the* logic of 'knowledge that' and KD45 is *the* logic of 'belief that'. While we adopt the latter, we start

KD5(B_i)	the principles of modal logic KD5 for B_i
S4(K_i)	the principles of modal logic S4 for K_i
KiB	$\mathsf{K}_i\varphi \rightarrow \mathsf{B}_i\varphi$
BiKB	$\mathsf{B}_i\varphi \rightarrow \mathsf{K}_i\mathsf{B}_i\varphi$
BiBK	$\mathsf{B}_i\varphi \rightarrow \mathsf{B}_i\mathsf{K}_i\varphi$
5(K_i)	$\neg\mathsf{K}_i\varphi \rightarrow \mathsf{K}_i\neg\mathsf{K}_i\varphi$

Table 1: Principles of the three logics EDL (first four lines), EDL+BiBK, and EDL+5(K_i).

from a weaker logic of knowledge, S4, and consider several extensions, including S5. Indeed, S5 was heavily criticised in the philosophical literature as being too strong a logic of knowledge [21, 23, 31, 15, 30].

In epistemology there is a long-standing debate about the relation between knowledge and belief. For long it was taken for granted that knowledge can be reduced to belief by defining ‘knowledge that φ’ as ‘justified true belief that φ’. However, Gettier’s counterexample showed that things are not so simple [14]; the subsequent debate is still ongoing and there is no consensus about whether such a reduction is possible and how it should be defined. A more cautious enterprise is to take both B_i and K_i as primitives and to study the interaction between these two modal operators. This however has to be done with care, as we will see later in this section.

The traditional language of epistemic-doxastic logic, here denoted by $\mathcal{L}$, is defined by the following grammar:

$$\mathcal{L} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{B}_i p \mid \mathsf{K}_i p$$

where p ranges over a countable set $\mathbf{P}$ of propositional variables and i ranges over a finite set $\mathbf{A}$ of agents. With the language $\mathcal{L}$ we are going to study three different epistemic-doxastic logics.

We suppose that the other boolean operators are defined as abbreviations: $\varphi \vee \psi$ abbreviates $\neg(\neg\varphi \wedge \neg\psi)$; $\varphi \rightarrow \psi$ abbreviates $\neg(\varphi \wedge \neg\psi)$; and $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We use the standard conventions for omitting parentheses; in particular, we suppose that all other operators bind stronger than $\leftrightarrow$.

The weakest logic EDL is axiomatised by the first four lines of Table 1, i.e., by KD5(B_i)+S4(K_i)+KiB+BiKB. The axiomatisation of KD5(B_i) can, for example, be made up of the axioms K(B_i): $\mathsf{B}_i(\varphi \rightarrow \psi) \rightarrow (\mathsf{B}_i\varphi \rightarrow \mathsf{B}_i\psi)$, D($\mathsf{B}_i$): $\mathsf{B}_i\varphi \rightarrow \neg\mathsf{B}_i\neg\varphi$, and 5($\mathsf{B}_i$): $\neg\mathsf{B}_i\varphi \rightarrow \mathsf{B}_i\neg\mathsf{B}_i\varphi$, and the inference rule of necessitation RN(B_i): ‘from φ infer $\mathsf{B}_i\varphi$’; and the axiomatisation of S4(K_i) of the axioms K(K_i): $\mathsf{K}_i(\varphi \rightarrow \psi) \rightarrow (\mathsf{K}_i\varphi \rightarrow \mathsf{K}_i\psi)$, 4($\mathsf{K}_i$): $\mathsf{K}_i\varphi \rightarrow \mathsf{K}_i\mathsf{K}_i\varphi$, and T($\mathsf{K}_i$): $\mathsf{K}_i\varphi \rightarrow \varphi$ and the inference rule of necessitation RN(K_i): ‘from φ infer $\mathsf{K}_i\varphi$’. The two remaining axioms are interaction axioms governing the interplay of knowledge and belief: KiB says that knowledge implies belief; BiKB is the axiom of positive introspection of belief.

Remark 1 *The positive introspection axiom 4(B_i): $\mathsf{B}_i\varphi \rightarrow \mathsf{B}_i\mathsf{B}_i\varphi$ is not in our list but can be derived from BiKB and KiB. Our logic of belief is therefore KD45(B_i).*

Similarly, the negative introspection axiom $\neg \mathsf{B}_i\varphi \to \mathsf{K}_i\neg\mathsf{B}_i\varphi$ *is not in our list but can be derived from KD5(*B_i*) and positive introspection of belief.*[1]

We will moreover be considering from Section 4 on two extensions of EDL the axioms of which are stated in the last two lines of Table 1. For the first, which we call EDL+BiBK, we add an axiom of strong belief BiBK to EDL: if i believes that φ then i believes that she knows that φ. (Therefore i cannot tell the difference between what she knows and what she merely believes.) The second, which we call EDL+5(K_i), does not feature the axiom BiBK, but has negative introspection for knowledge instead.

Remark 2 *The strong belief axiom BiBK makes the following equivalences provable in EDL+BiBK:*

$$\mathsf{K}_i(\mathsf{B}_i\varphi \to \mathsf{K}_i\psi) \leftrightarrow \mathsf{B}_i\varphi \to \mathsf{K}_i\psi,$$
$$\mathsf{K}_i(\varphi \to \mathsf{B}_i\psi) \leftrightarrow \mathsf{K}_i\neg\varphi \vee \mathsf{B}_i\psi,$$
$$\mathsf{K}_i(\varphi \to \neg\mathsf{B}_i\psi) \leftrightarrow \mathsf{K}_i\neg\varphi \vee \neg\mathsf{B}_i\psi.$$

Remark 3 *It is known that the extension of EDL by both BiBK and 5(*K_i*) is not very interesting: in that logic belief implies knowledge [24]. To see that, it suffices to prove that* $(\mathsf{B}_i\varphi \wedge \neg\mathsf{K}_i\varphi) \to \bot$ *becomes a theorem.*[2] *We refer the reader to the work of Voorbraak for further results and discussions [31, 15].*

It was established by Lenzen that the logic EDL+BiBK is strongly related to the modal logic S4.2.

Proposition 4 ([22]) *The equivalence* $\mathsf{B}_i\varphi \leftrightarrow \neg\mathsf{K}_i\neg\mathsf{K}_i\varphi$ *is a theorem of EDL+BiBK. The logic S4.2 together with the axiom* $\mathsf{B}_i\varphi \leftrightarrow \neg\mathsf{K}_i\neg\mathsf{K}_i\varphi$ *is an equivalent axiomatisation of EDL+BiBK.*

Beyond Lenzen's papers, an excellent survey of extensions of EDL and their properties can be found in Aucher's papers [2] and [3]. (The second publication extends the conference version in [1].)

Proposition 4 allows us to settle the complexity of deciding provability of a formula φ in EDL+BiBK. If we replace all subformulas $\mathsf{B}_i\psi$ of φ by $\neg\mathsf{K}_i\neg\mathsf{K}_i\psi$ then the resulting formula φ' only contains modal operators K_i, and its length is linear in the length of the original φ. Following Proposition 4, φ' is equivalent to φ in EDL+BiBK. Moreover and again by Proposition 4, φ' is a theorem of EDL+BiBK if and only if φ' is a theorem of

[1]First, $\neg\mathsf{B}_i\varphi \to \mathsf{B}_i\neg\mathsf{B}_i\varphi$ by 5(B_i); second, $\mathsf{B}_i\neg\mathsf{B}_i\varphi \to \mathsf{K}_i\mathsf{B}_i\neg\mathsf{B}_i\varphi$ by the positive introspection axiom BiKB; third, $\mathsf{K}_i\mathsf{B}_i\neg\mathsf{B}_i\varphi \to \mathsf{K}_i\neg\mathsf{B}_i\varphi$ by 4(B_i), D(B_i), and because K_i is a normal modal operator.

[2]A proof is:

1.	$\mathsf{B}_i\varphi \to \mathsf{B}_i\mathsf{K}_i\varphi$	BiBK
2.	$\neg\mathsf{K}_i\varphi \to \mathsf{K}_i\neg\mathsf{K}_i\varphi$	5(K_i)
3.	$\mathsf{K}_i\neg\mathsf{K}_i\varphi \to \mathsf{B}_i\neg\mathsf{K}_i\varphi$	KiB
4.	$(\mathsf{B}_i\mathsf{K}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{K}_i\varphi) \to \bot$	KD(B_i)
5.	$(\mathsf{B}_i\varphi \wedge \neg\mathsf{K}_i\varphi) \to \bot$	from 1,2,3,4

S4.2. The results in [29, 7, 8] that the problem of deciding provability in S4.2 is PSPACE-complete can be generalised to multi-agent S4.2, as confirmed in [9] (an unpublished extended version of [7]). Hence both provability and consistency in EDL+BiBK are PSPACE-complete. We do not know whether complexity results for the logics EDL and EDL+5(K_i) exist; we conjecture that provability is PSPACE-complete in these logics as well.

3 'Belief-About' and Epistemic-Doxastic Situations

In our language of 'knowledge that' and 'belief that' we can define several modal operators of the kind 'having a belief about a proposition' and 'having knowledge about a proposition'. We show that the modalities 'true belief about' and 'mere belief about' play a particular role: they allow us to define in a combinatorial way all possible epistemic-doxastic situations about a contingent proposition φ. Moreover, we show how 'knowledge that' and 'belief that' can be defined from the modalities 'true belief about' and 'mere belief about'.

3.1 From 'Belief-That' to 'Belief-About'

Let us define the following modalities as abbreviations in the language $\mathcal{L}$:

$$\mathsf{BA}_i\varphi = \mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi \qquad \text{“}i\text{ has a belief about }\varphi\text{”}$$

$$\mathsf{KA}_i\varphi = \mathsf{K}_i\varphi \vee \mathsf{K}_i\neg\varphi \qquad \text{“}i\text{ has knowledge about }\varphi\text{”}$$

$$\mathsf{TBA}_i\varphi = (\varphi \wedge \mathsf{B}_i\varphi) \vee (\neg\varphi \wedge \mathsf{B}_i\neg\varphi) \qquad \text{“}i\text{ has a true belief about }\varphi\text{”}$$

$$\mathsf{MBA}_i\varphi = (\mathsf{B}_i\varphi \wedge \neg\mathsf{K}_i\varphi) \vee (\mathsf{B}_i\neg\varphi \wedge \neg\mathsf{K}_i\neg\varphi) \qquad \text{“}i\text{ has a mere belief about }\varphi\text{”}$$

Observe that by principles of propositional logic, $\mathsf{TBA}_i\varphi$ is equivalent to $(\varphi \to \mathsf{B}_i\varphi) \wedge (\neg\varphi \to \mathsf{B}_i\neg\varphi)$. Moreover, $\mathsf{MBA}_i\varphi$ is equivalent to $(\mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi) \wedge \neg\mathsf{K}_i\varphi \wedge \neg\mathsf{K}_i\neg\varphi$ and hence to $\mathsf{BA}_i\varphi \wedge \neg\mathsf{KA}_i\varphi$. (The first equivalence holds thanks to axioms KiB and D(B_i).)

Proposition 5 *The following equivalences hold:*

$$\mathsf{BA}_i\neg\varphi \leftrightarrow \mathsf{BA}_i\varphi, \qquad \mathsf{TBA}_i\neg\varphi \leftrightarrow \mathsf{TBA}_i\varphi,$$

$$\mathsf{KA}_i\neg\varphi \leftrightarrow \mathsf{KA}_i\varphi, \qquad \mathsf{MBA}_i\neg\varphi \leftrightarrow \mathsf{MBA}_i\varphi.$$

3.2 A 'Belief-About' Fragment of the Language $\mathcal{L}$

Let $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$ be the fragment of the language $\mathcal{L}$ where the only modal operators are TBA_i and MBA_i. Hence the grammar of $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$ is:

$$\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{TBA}_i\varphi \mid \mathsf{MBA}_i\varphi.$$

In the rest of the paper we are going to investigate the properties of $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$. In the rest of the present section we show that it has enough expressivity to account for all possible epistemic situations about a contingent proposition and to capture the 'knowledge that' and 'belief that' modalities. In the next section we investigate whether and how consecutive modalities can be reduced.

3.3 First-Order Epistemic-Doxastic Situations

Let φ be a contingent formula, i.e., a formula such that both φ and $\neg\varphi$ are consistent. There are eight possible epistemic-doxastic situations that can be expressed in the traditional language of epistemic logic $\mathcal{L}$, namely:

$\varphi \wedge \mathsf{K}_i\varphi$	$\neg\varphi \wedge \mathsf{K}_i\neg\varphi$
$\varphi \wedge \mathsf{B}_i\varphi \wedge \neg\mathsf{K}_i\varphi$	$\neg\varphi \wedge \mathsf{B}_i\neg\varphi \wedge \neg\mathsf{K}_i\neg\varphi$
$\varphi \wedge \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi$	$\neg\varphi \wedge \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi$
$\varphi \wedge \mathsf{B}_i\neg\varphi$	$\neg\varphi \wedge \mathsf{B}_i\varphi$

These distinctions can not only be expressed in $\mathcal{L}$, but also in the fragment $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$.

Proposition 6 *The following equivalences are theorems of EDL:*

$$\varphi \wedge \mathsf{K}_i\varphi \leftrightarrow \varphi \wedge \mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi,$$
$$\neg\varphi \wedge \mathsf{K}_i\neg\varphi \leftrightarrow \neg\varphi \wedge \mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi,$$
$$\varphi \wedge \mathsf{B}_i\varphi \wedge \neg\mathsf{K}_i\varphi \leftrightarrow \varphi \wedge \mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi,$$
$$\neg\varphi \wedge \mathsf{B}_i\neg\varphi \wedge \neg\mathsf{K}_i\neg\varphi \leftrightarrow \neg\varphi \wedge \mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi,$$
$$\varphi \wedge \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi \leftrightarrow \varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi,$$
$$\neg\varphi \wedge \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi \leftrightarrow \neg\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi,$$
$$\varphi \wedge \mathsf{B}_i\neg\varphi \leftrightarrow \varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi,$$
$$\neg\varphi \wedge \mathsf{B}_i\varphi \leftrightarrow \neg\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi.$$

Proof. These equivalences can be proved with the KiB axiom together with principles of normal modal logics. The proof amounts to spelling out the definitions of TBA_i and MBA_i and applying the axioms KiB, K(B_i), K(K_i) and the inference rules RN(B_i) and RN(K_i). For example, the first equivalence can be proved as follows:

$$\begin{aligned}\varphi \wedge \mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi &\leftrightarrow \varphi \wedge \mathsf{B}_i\varphi \wedge (\mathsf{B}_i\varphi \to \mathsf{K}_i\varphi)\\ &\leftrightarrow \varphi \wedge \mathsf{B}_i\varphi \wedge \mathsf{K}_i\varphi\\ &\leftrightarrow \varphi \wedge \mathsf{K}_i\varphi \qquad \text{by KiB}\end{aligned}$$

The other proofs are similar. ■

Thanks to Proposition 6, the eight possible epistemic-doxastic situations can also be characterised in the fragment $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$:

$\varphi \wedge \mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi$	$\neg\varphi \wedge \mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi$
$\varphi \wedge \mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi$	$\neg\varphi \wedge \mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi$
$\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi$	$\neg\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi$
$\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi$	$\neg\varphi \wedge \neg\mathsf{TBA}_i\varphi \wedge \mathsf{MBA}_i\varphi$

Hence we have characterised all possible epistemic-doxastic situations in terms of three independent components. This can be compared to the study of different normative positions in deontic logic as initiated by Kanger and Lindahl and studied more recently by Sergot [28].

3.4 Higher-Order Epistemic-Doxastic Situations

We can generalise first-order epistemic-doxastic situations to higher orders. Let us demonstrate this by going through Sally's second-order epistemic-doxastic situations, i.e., her beliefs about Ann's beliefs.

1. Ann puts the marble in the basket. Sally then knows that Ann knows that the marble is in the basket, i.e., she has a belief about Ann's beliefs that is both true and not a mere belief:

$$b \wedge \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_A b \wedge$$
$$\mathsf{TBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_S\, \mathsf{TBA}_A b \wedge$$
$$\neg\mathsf{MBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{MBA}_A b \wedge \neg\mathsf{MBA}_S\, \mathsf{MBA}_A b.$$

2. Sally leaves the room. Sally continues to believe that Ann sees the marble, but she no longer knows that:

$$b \wedge \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_A b \wedge$$
$$\mathsf{TBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{TBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{TBA}_A b \wedge$$
$$\mathsf{MBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{MBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{MBA}_A b.$$

3. Ann transfers the marble to the box. Sally's belief that Ann knows where the marble is remains true:

$$\neg b \wedge \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_A b \wedge$$
$$\neg\mathsf{TBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{TBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{TBA}_A b \wedge$$
$$\mathsf{MBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{MBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{MBA}_A b.$$

4. Sally re-enters and looks inside the basket. We consider several possibilities as to the evolution of her beliefs about Ann's beliefs:

   (a) Sally and Ann look into the basket together. Now Sally knows that Ann also knows that the marble isn't there:

$$\neg b \wedge \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_A b \wedge$$
$$\mathsf{TBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_S\, \mathsf{TBA}_A b \wedge$$
$$\neg\mathsf{MBA}_S\, b \wedge \mathsf{TBA}_S\, \mathsf{MBA}_A b \wedge \neg\mathsf{MBA}_S\, \mathsf{MBA}_A b.$$

   (b) Sally looks into the basket without Ann. She may then believe that Ann still thinks that the marble is in the basket when it is not. Therefore Sally's beliefs about Ann's beliefs become untrue:

$$\neg b \wedge \mathsf{TBA}_A b \wedge \neg\mathsf{MBA}_A b \wedge$$
$$\mathsf{TBA}_S\, b \wedge \neg\mathsf{TBA}_S\, \mathsf{TBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{TBA}_A b \wedge$$
$$\neg\mathsf{MBA}_S\, b \wedge \neg\mathsf{TBA}_S\, \mathsf{MBA}_A b \wedge \mathsf{MBA}_S\, \mathsf{MBA}_A b.$$

(c) Sally looks into the basket without Ann. She sees that the marble is not there, but believes that Ann is the one who took it out, and therefore that Ann still knows the location of the marble. As long as she has no confirmation, this remains a mere belief:

$$\neg b \land \mathsf{TBA}_A b \land \neg \mathsf{MBA}_A b \land$$
$$\neg \mathsf{TBA}_S b \land \mathsf{TBA}_S \mathsf{TBA}_A b \land \mathsf{MBA}_S \mathsf{TBA}_A b \land$$
$$\mathsf{MBA}_S b \land \mathsf{TBA}_S \mathsf{MBA}_A b \land \mathsf{MBA}_S \mathsf{MBA}_A b.$$

3.5 From ‘Belief-About’ to ‘Belief-That’

The definition of ‘about’ modalities from ‘that’ modalities of Section 3 is straightforward. We now consider the other way around: expressing ‘that’ modalities using ‘about’ modalities. It is known that this can be done for ‘knowledge about’: thanks to the truth axiom T(K_i), the formula $\mathsf{K}_i\varphi$ is equivalent to $\varphi \land \mathsf{KA}_i\varphi$. It is also known that the ‘belief about’ modality alone cannot express the belief-that modality [13]. We show now that the fragment $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$ is fully expressive: together, the two modalities of true belief TBA_i and of mere belief MBA_i are enough to express ‘belief that’ and ‘knowledge that’.

Proposition 7 *The following equivalences are theorems of EDL:*

$$\mathsf{KA}_i\varphi \leftrightarrow \mathsf{TBA}_i\varphi \land \neg\mathsf{MBA}_i\varphi,$$
$$\mathsf{K}_i\varphi \leftrightarrow \varphi \land \mathsf{TBA}_i\varphi \land \neg\mathsf{MBA}_i\varphi,$$
$$\mathsf{BA}_i\varphi \leftrightarrow \mathsf{TBA}_i\varphi \lor \mathsf{MBA}_i\varphi,$$
$$\mathsf{B}_i\varphi \leftrightarrow (\varphi \land \mathsf{TBA}_i\varphi) \lor (\neg\varphi \land \neg\mathsf{TBA}_i\varphi \land \mathsf{MBA}_i\varphi).$$

Proof. This follows from Proposition 6. ■

4 Reduction of ‘About’ Modalities

In this section we explore the interplay between the different modalities, as governed in particular by principles of introspection. We begin by listing some equivalences of the base logic EDL, then we investigate some more properties of its two extensions EDL+BiBK and EDL+5(K_i).

4.1 Properties of EDL

Our first group of equivalences is about traditional operators followed by TBA_i and MBA_i and their negations.

Proposition 8 *The following equivalences are theorems of EDL:*

$$\mathsf{B}_i\mathsf{TBA}_i\varphi \leftrightarrow \mathsf{BA}_i\varphi, \qquad \mathsf{K}_i\mathsf{TBA}_i\varphi \leftrightarrow \mathsf{KA}_i\varphi,$$
$$\mathsf{B}_i\neg\mathsf{TBA}_i\varphi \leftrightarrow \neg\mathsf{BA}_i\varphi, \qquad \mathsf{K}_i\neg\mathsf{TBA}_i\varphi \leftrightarrow \neg\mathsf{BA}_i\varphi,$$
$$\mathsf{BA}_i\mathsf{TBA}_i\varphi \leftrightarrow \top.$$

Proof. We first prove the equivalences of the first column.

$$\begin{aligned}
\mathsf{B}_i\mathsf{TBA}_i\varphi &\leftrightarrow \mathsf{B}_i(\neg\varphi \vee \mathsf{B}_i\varphi) \wedge \mathsf{B}_i(\varphi \vee \mathsf{B}_i\neg\varphi) \\
&\leftrightarrow (\mathsf{B}_i\neg\varphi \vee \mathsf{B}_i\varphi) \wedge (\mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi) && \text{by KD45}(\mathsf{B}_i) \\
&\leftrightarrow \mathsf{BA}_i\varphi \\
\mathsf{B}_i\neg\mathsf{TBA}_i\varphi &\leftrightarrow \mathsf{B}_i\neg((\varphi \wedge \mathsf{B}_i\varphi) \vee (\neg\varphi \wedge \mathsf{B}_i\neg\varphi)) \\
&\leftrightarrow \mathsf{B}_i(\neg\varphi \vee \neg\mathsf{B}_i\varphi) \wedge \mathsf{B}_i(\varphi \vee \neg\mathsf{B}_i\neg\varphi) \\
&\leftrightarrow (\mathsf{B}_i\neg\varphi \vee \neg\mathsf{B}_i\varphi) \wedge (\mathsf{B}_i\varphi \vee \neg\mathsf{B}_i\neg\varphi) && \text{by KD45}(\mathsf{B}_i) \\
&\leftrightarrow \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi && \text{by KD}(\mathsf{B}_i) \\
&\leftrightarrow \neg\mathsf{BA}_i\varphi
\end{aligned}$$

The last equivalence of the first column follows immediately from the two above results.

We now move on to the second column.

$$\begin{aligned}
\mathsf{K}_i\mathsf{TBA}_i\varphi &\leftrightarrow \mathsf{K}_i((\varphi \rightarrow \mathsf{B}_i\varphi) \wedge (\neg\varphi \rightarrow \mathsf{B}_i\neg\varphi)) \\
&\leftrightarrow \mathsf{K}_i(\varphi \rightarrow \mathsf{B}_i\varphi) \wedge \mathsf{K}_i(\neg\varphi \rightarrow \mathsf{B}_i\neg\varphi) \\
&\leftrightarrow (\mathsf{K}_i\neg\varphi \vee \mathsf{B}_i\varphi) \wedge (\mathsf{K}_i\varphi \vee \mathsf{B}_i\neg\varphi) && \text{by Remark 1} \\
&\leftrightarrow \mathsf{K}_i\neg\varphi \vee \mathsf{K}_i\varphi \\
&\leftrightarrow \mathsf{KA}_i\varphi \\
\mathsf{K}_i\neg\mathsf{TBA}_i\varphi &\leftrightarrow \mathsf{K}_i\neg((\varphi \wedge \mathsf{B}_i\varphi) \vee (\neg\varphi \wedge \mathsf{B}_i\neg\varphi)) \\
&\leftrightarrow \mathsf{K}_i(\neg\varphi \vee \neg\mathsf{B}_i\varphi) \wedge \mathsf{K}_i(\varphi \vee \neg\mathsf{B}_i\neg\varphi) \\
&\leftrightarrow (\mathsf{K}_i\neg\varphi \vee \neg\mathsf{B}_i\varphi) \wedge (\mathsf{K}_i\varphi \vee \neg\mathsf{B}_i\neg\varphi) && \text{by Remark 1} \\
&\leftrightarrow \neg\mathsf{B}_i\varphi \wedge \neg\mathsf{B}_i\neg\varphi \\
&\leftrightarrow \neg\mathsf{BA}_i\varphi
\end{aligned}$$

This ends the proof. ■

The above equivalences allow us to reduce consecutive modalities $\mathsf{TBA}_i\mathsf{TBA}_i$ and $\mathsf{MBA}_i\mathsf{TBA}_i$.

Proposition 9 *The following equivalences hold in EDL:*

$$\mathsf{TBA}_i\mathsf{TBA}_i\varphi \leftrightarrow \mathsf{TBA}_i\varphi \vee \neg\mathsf{MBA}_i\varphi,$$
$$\mathsf{MBA}_i\mathsf{TBA}_i\varphi \leftrightarrow \mathsf{MBA}_i\varphi.$$

Proof. For the first equivalence:

$$\begin{aligned}
\mathsf{TBA}_i\mathsf{TBA}_i\varphi &\leftrightarrow (\mathsf{TBA}_i\varphi \wedge \mathsf{B}_i\mathsf{TBA}_i\varphi) \vee (\neg\mathsf{TBA}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{TBA}_i\varphi) \\
&\leftrightarrow (\mathsf{TBA}_i\varphi \wedge \mathsf{BA}_i\varphi) \vee (\neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{BA}_i\varphi) && \text{by Proposition 8} \\
&\leftrightarrow \mathsf{TBA}_i\varphi \vee (\neg\mathsf{TBA}_i\varphi \wedge \neg\mathsf{MBA}_i\varphi) && \text{by Proposition 7} \\
&\leftrightarrow \mathsf{TBA}_i\varphi \vee \neg\mathsf{MBA}_i\varphi
\end{aligned}$$

For the second equivalence:

$$\begin{aligned} \mathsf{MBA}_i\mathsf{TBA}_i\varphi &\leftrightarrow \mathsf{BA}_i\mathsf{TBA}_i\varphi \wedge \neg\mathsf{K}_i\mathsf{TBA}_i\varphi \wedge \neg\mathsf{K}_i\neg\mathsf{TBA}_i\varphi \\ &\leftrightarrow \top \wedge \neg\mathsf{KA}_i\varphi \wedge \mathsf{BA}_i\varphi && \text{by Proposition 8} \\ &\leftrightarrow \mathsf{MBA}_i\varphi \end{aligned}$$

This ends the proof. ■

We conjecture that the logic EDL is not strong enough to allow us to reduce consecutive modalities $\mathsf{TBA}_i\mathsf{MBA}_i$ and $\mathsf{MBA}_i\mathsf{MBA}_i$. We do not establish this formally and show instead that such reductions exist for the two extensions of EDL, EDL+BiBK and EDL+5(K_i).

4.2 Properties of EDL+BiBK

We once again begin by investigating the interactions between traditional operators and TBA_i and MBA_i.

Proposition 10 *The following equivalences hold in EDL+BiBK:*

$$\begin{aligned} \mathsf{B}_i\mathsf{MBA}_i\varphi &\leftrightarrow \bot, & \mathsf{K}_i\mathsf{MBA}_i\varphi &\leftrightarrow \bot, \\ \mathsf{B}_i\neg\mathsf{MBA}_i\varphi &\leftrightarrow \top, & \mathsf{K}_i\neg\mathsf{MBA}_i\varphi &\leftrightarrow \neg\mathsf{MBA}_i\varphi. \end{aligned}$$

Proof. We start with the left column.

$$\begin{aligned} \mathsf{B}_i\mathsf{MBA}_i\varphi &\leftrightarrow \mathsf{B}_i((\mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi) \wedge \neg\mathsf{K}_i\varphi \wedge \neg\mathsf{K}_i\neg\varphi) \\ &\leftrightarrow \mathsf{B}_i(\mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi) \wedge \mathsf{B}_i\neg\mathsf{K}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{K}_i\neg\varphi \\ &\leftrightarrow (\mathsf{B}_i\varphi \vee \mathsf{B}_i\neg\varphi) \wedge \mathsf{B}_i\neg\mathsf{K}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{K}_i\neg\varphi \\ &\leftrightarrow (\mathsf{B}_i\mathsf{K}_i\varphi \vee \mathsf{B}_i\mathsf{K}_i\neg\varphi) \wedge \mathsf{B}_i\neg\mathsf{K}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{K}_i\neg\varphi && \text{by BiBK} \\ &\leftrightarrow \bot \\ \mathsf{B}_i\neg\mathsf{MBA}_i\varphi &\leftrightarrow \mathsf{B}_i((\mathsf{B}_i\varphi \to \mathsf{K}_i\varphi) \wedge (\mathsf{B}_i\neg\varphi \to \mathsf{K}_i\neg\varphi)) \\ &\leftrightarrow \mathsf{B}_i(\mathsf{B}_i\varphi \to \mathsf{K}_i\varphi) \wedge \mathsf{B}_i(\mathsf{B}_i\neg\varphi \to \mathsf{K}_i\neg\varphi) \\ &\leftrightarrow (\mathsf{B}_i\varphi \to \mathsf{B}_i\mathsf{K}_i\varphi) \wedge (\mathsf{B}_i\neg\varphi \to \mathsf{B}_i\mathsf{K}_i\neg\varphi) && \text{by KD45}(\mathsf{B}_i) \\ &\leftrightarrow \top && \text{by BiBK} \end{aligned}$$

It remains to prove the right column.

$$\begin{aligned} \mathsf{K}_i\mathsf{MBA}_i\varphi &\to \mathsf{B}_i\mathsf{MBA}_i\varphi \\ &\to \bot \\ \mathsf{K}_i\neg\mathsf{MBA}_i\varphi &\leftrightarrow \mathsf{K}_i((\mathsf{B}_i\varphi \to \mathsf{K}_i\varphi) \wedge (\mathsf{B}_i\neg\varphi \to \mathsf{K}_i\neg\varphi)) \\ &\leftrightarrow (\mathsf{B}_i\varphi \to \mathsf{K}_i\varphi) \wedge (\mathsf{B}_i\neg\varphi \to \mathsf{K}_i\neg\varphi) && \text{by Remark 1} \\ &\leftrightarrow \neg\mathsf{MBA}_i\varphi \end{aligned}$$

■

Together with Proposition 9, the next result establishes that in EDL+BiBK all combinations of TBA_i and MBA_i can be reduced:

Proposition 11 *The following equivalences hold in EDL+BiBK:*

$$\mathsf{TBA}_i\mathsf{MBA}_i\varphi \leftrightarrow \neg\mathsf{MBA}_i\varphi,$$
$$\mathsf{MBA}_i\mathsf{MBA}_i\varphi \leftrightarrow \mathsf{MBA}_i\varphi.$$

Proof. The proof makes use of Proposition 10:

$$\begin{aligned}
\mathsf{TBA}_i\mathsf{MBA}_i\varphi &\leftrightarrow (\mathsf{MBA}_i\varphi \wedge \mathsf{B}_i\mathsf{MBA}_i\varphi) \vee (\neg\mathsf{MBA}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{MBA}_i\varphi)\\
&\leftrightarrow (\mathsf{MBA}_i\varphi \wedge \bot) \vee (\neg\mathsf{MBA}_i\varphi \wedge \top)\\
&\leftrightarrow \neg\mathsf{MBA}_i\varphi\\
\mathsf{MBA}_i\mathsf{MBA}_i\varphi &\leftrightarrow (\mathsf{B}_i\mathsf{MBA}_i\varphi \wedge \neg\mathsf{K}_i\mathsf{MBA}_i\varphi) \vee (\mathsf{B}_i\neg\mathsf{MBA}_i\varphi \wedge \neg\mathsf{K}_i\neg\mathsf{MBA}_i\varphi)\\
&\leftrightarrow (\bot \wedge \neg\bot) \vee (\top \wedge \neg\neg\mathsf{MBA}_i\varphi)\\
&\leftrightarrow \mathsf{MBA}_i\varphi
\end{aligned}$$

■

4.3 Properties of EDL+5(K_i)

We show once again that all combinations of TBA_i and MBA_i can be reduced, now considering EDL+5(K_i). In this logic the reductions are quite straightforward.

Proposition 12 *The following equivalences hold in EDL+5(K_i):*

$$\mathsf{KA}_i\mathsf{MBA}_i\varphi \leftrightarrow \top,$$
$$\mathsf{MBA}_i\mathsf{MBA}_i\varphi \leftrightarrow \bot,$$
$$\mathsf{TBA}_i\mathsf{MBA}_i\varphi \leftrightarrow \top.$$

Proof. The introspective principles tell us that $\mathsf{MBA}_i\varphi \rightarrow \mathsf{K}_i\mathsf{MBA}_i\varphi$ and $\neg\mathsf{MBA}_i\varphi \rightarrow \mathsf{K}_i\neg\mathsf{MBA}_i\varphi$, hence the first equivalence. From there, we can show:

$$\begin{aligned}
\mathsf{MBA}_i\mathsf{MBA}_i\varphi &\leftrightarrow \mathsf{BA}_i\mathsf{MBA}_i\varphi \wedge \neg\mathsf{KA}_i\mathsf{MBA}_i\varphi\\
&\leftrightarrow \mathsf{BA}_i\mathsf{MBA}_i\varphi \wedge \bot\\
&\leftrightarrow \bot
\end{aligned}$$

Following the same reasoning as for the first equivalence, we also have that $\mathsf{MBA}_i\varphi \rightarrow \mathsf{B}_i\mathsf{MBA}_i\varphi$ and $\neg\mathsf{MBA}_i\varphi \rightarrow \mathsf{B}_i\neg\mathsf{MBA}_i\varphi$. Therefore:

$$\begin{aligned}
\mathsf{TBA}_i\mathsf{MBA}_i\varphi &\leftrightarrow (\mathsf{MBA}_i\varphi \wedge \mathsf{B}_i\mathsf{MBA}_i\varphi) \vee (\neg\mathsf{MBA}_i\varphi \wedge \mathsf{B}_i\neg\mathsf{MBA}_i\varphi)\\
&\leftrightarrow \mathsf{MBA}_i\varphi \vee \neg\mathsf{MBA}_i\varphi\\
&\leftrightarrow \top
\end{aligned}$$

■

These last two reductions follow the intuition that in EDL+5(K_i) the agents can always tell whether or not their beliefs are based on knowledge, whereas in EDL+BiBK agents do not consider the possibility that their beliefs are mere beliefs.

5 Conclusion

We have studied two 'belief about' modalities in the framework of three epistemic-doxastic logics whose base logic combines KD45 for the 'belief that' modality and S4 for the 'knowledge that' modality. The latter is actually a S4.2 modality if we assume the 'belief implies belief to know' axiom BiBK; moreover, in that case the belief modality reduces to two consecutive knowledge modalities. Our 'true belief about' and 'mere belief about' modalities can express in a combinatorial way all eight possible epistemic situations. They are also expressive enough to capture the 'belief that' and 'knowledge that' modalities.

For all three logics, an axiomatisation of the theorems in the language $\mathcal{L}_{\mathsf{TBA},\mathsf{MBA}}$ can be obtained in a very simple manner, namely by taking the traditional axiomatisation of Section 2 and substituting K_i and B_i by their definitions in terms of TBA_i and MBA_i of Section 3.5. This is straightforward, but we do not find this very informative because the resulting axioms are complicated, particularly as the modal operators TBA_i and MBA_i neither satisfy the monotony axioms $\mathsf{TBA}_i(\varphi \land \psi) \to (\mathsf{TBA}_i\varphi \land \mathsf{TBA}_i\psi)$ and $\mathsf{MBA}_i(\varphi \land \psi) \to (\mathsf{MBA}_i\varphi \land \mathsf{MBA}_i\psi)$, nor the conjunction axioms $(\mathsf{TBA}_i\varphi \land \mathsf{TBA}_i\psi) \to \mathsf{TBA}_i(\varphi \land \psi)$ and $(\mathsf{MBA}_i\varphi \land \mathsf{MBA}_i\psi) \to \mathsf{MBA}_i(\varphi \land \psi)$.

One of the perspectives for future work is the definition of lightweight fragments of our language, as previously done for the standard modal operators B_i [27] and K_i [17, 10]. In these papers, in order to decrease complexity of epistemic reasoning the language is restricted to boolean combinations of what may be called epistemic atoms. The definition of the latter varies depending on the paper, but they are typically sequences of modal operators (and possibly negations) followed by propositional variables. In the same spirit we can define *epistemic-doxastic atoms* as sequences of modalities TBA_i and MBA_i that are followed by a propositional variable: they are of the form

$$M_1 \cdots M_d p$$

where $p \in \mathbf{P}$ and where M_k is either TBA_i or MBA_i, for some i. While these fragments are less expressive than the entire language, they typically enjoy lower complexity of the provability problem, which makes them interesting for knowledge representation. We have already exploited this in previous work about the 'knowledge about' modality, where we have established that provability is coNP-complete [17, 18, 11]. This enabled the reduction of epistemic planning tasks to classical planning tasks [10]. We have seen in the introduction how the evolution of Sally's belief in the Sally-Ann Test naturally corresponds to atomic modifications of the 'true belief about' and 'mere belief about' modalities. This can be compared to the modelisation of the Sally-Ann Test in Dynamic Epistemic Logic [6], which is a rich framework with high complexity; in particular, the plan existence decision problem is known to be undecidable [5]. We therefore expect lightweight epistemic-doxastic logic to be able to account for the evolution of belief and knowledge in applications where autonomous agents have to be equipped with a theory of mind in order to reason about other agents.

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