Prior and Łukasiewicz on Modal Logic

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Abstract

A. N. Prior was strongly influenced by the work of Polish logicians, especially Jan Łukasiewicz. One important consequence is his adoption of Łukasiewicz's bracket-free notation for logical formulae, but he also took issue with Łukasiewicz's criticism of Aristotle's views on possibility. The present paper looks at the rôle of I. M. Bocheński in making Prior aware of the Polish logical tradition.

1 Łukasiewicz's notation for propositional logic

One of the things A.N. Prior is noted for is his espousal of the logical notation devised by Jan Łukasiewicz. Although Prior preferred to use Łukasiewicz' notation, this symbolism has become less common over the years, and some explanation may be advisable. For those unfamiliar with what is often called 'Polish' notation I give here a glossary of the terminology. Where α and β are any well−formed formulae (wff) of a formal language incorporating propositional logic we have the following symbolism:[1]

Name	English	Łukasiewicz	Russell
Negation	Not	$N\alpha$	$\sim \alpha$
Disjunction	Either α or β	$A\alpha\beta$	$(\alpha \vee \beta)$
Conjunction	Both α and β	$K\alpha\beta$	$(\alpha \wedge \beta)$
Implication	If α then β	$C\alpha\beta$	$(\alpha \supset \beta)$
Equivalence	α if and only if β	$E\alpha\beta$	$(\alpha \equiv \beta)$

In order to illustrate how this notation works I will shew how it deals with scope ambiguities. Consider the two formulae $(\sim p \supset q)$ and $\sim(p \supset q)$. These wff have obviously different meanings, and the parentheses make it clear. That is why the formation rules have to say that if α and β are wff then so is $(\alpha \supset \beta)$. But it is a common convention to leave off the outermost parentheses, and write simply $\sim p \supset q$ rather than $(\sim p \supset q)$. Of course in $\sim(p \supset q)$ the parentheses cannot be dropped, since dropping them would leave you with $\sim p \supset q$, the same formula as before. In the case of the Łukasiewicz notation the matter is handled differently. The two wff are *CNpq* and *NCpq*, where the

[1] Whitehead and Russell [18] used a dot for conjunction rather than $\wedge$. and others have used &. In the case of implication $\rightarrow$ is often used in place of $\supset$. In predicate logic Łukasiewicz uses Σ and Π for the existential and universal quantifiers. Thus, in place of $\exists x$ and $\forall x$ you will find Σx and Πx. (Of course $\exists x$ and $\forall x$ are not strictly that of [18], which, for instance, uses $(\exists x)$ and (x) and a complicated system of dots in place of parentheses.)

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difference is signalled by the fact that the main operator is at the beginning, and in the former case it is *C* and in the latter case it is *N*.[2]

It is no part of my claim that the Łukasiewicz notation is superior (or inferior) to Russell's. My point in this paper is to look at what might have motivated one of the few authors who was not Polish to adopt the Łukasiewicz notation, and to champion it throughout the English-speaking world. It turns out that Prior's interest in Łukasiewicz has as much to do with issues in modal logic as it does with the question of notation.

2 Łukasiewicz and modal logic

In Prior's case one factor in his paying attention to Łukasiewicz's work is Łukasiewicz's criticism of Aristotle's view of possibility, a criticism presented in [2], I.M. Bocheński's 1947 book on the modal logic of Theophrastus, Aristotle's successor at the Lyceum. In the preface, on p. 5. of [2] Bocheński speaks of his 1947 book as presenting work directed by Łukasiewicz at the University of Warsaw in 1937. On p. 12 he summarises the Łukasiewicz notation mentioned above. He then introduces the two modal operators, *M* for possibility, as has now become common as an alternative to $\diamond$, and *S* for necessity:

> En outre, nous utiliserons les symbols suivants:
> *Mp* il est possible que *p*
> *Sp* il est nécessaire que *p*
> *Zp* *p* : ≪*Z*≫ ne sert qu'à rappeler que ≪*p*≫ n'est qualifié ni de ≪*M*≫, ni de ≪*S*≫. ([2], p. 12)

This suggests that the modal operators are not Łukasiewicz's, and in fact Łukasiewicz was not a friend of modal logic; principally because he thought it led Aristotle into error. Aristotle, it seems, argued that possibility is incompatible with necessity. That is to say, in terms of Bocheński's symbol *M*, (but using Russell's notation for the truth functors), Aristotle believed that *Mp* is equivalent to *M*~*p*.[3]

Theophrastus argued that, in contrast to Aristotle's view, the necessary should include the possible rather than be incompatible with it. One feature of Bocheński's book

[2]It is unlikely that the Russell notation would be confusing in this case, but there can be cases which may lead to confusion. Consider a case in predicate logic where, provided x is a variable not free in β we have $\exists x(\alpha \supset \beta) \equiv (\forall x\alpha \supset \beta)$. In particular the instance $\exists x(\varphi x \supset p) \equiv (\forall x\varphi x \supset p)$, where p contains no free variables, is valid; so that if φ means 'is successfully answered' and p means 'the candidate obtains full marks', then the equivalence means that (in a domain consisting of a set of questions attempted by a given candidate) the formula could plausibly be read as stating the equivalence between 'If the candidate successfully answers every question then the candidate will obtain full marks' and 'there is a question such that if the candidate successfully answers that question then the candidate obtains full marks'. Whatever may be said about this equivalence it should be clear that the parentheses are crucial, since there is no equivalence between $(\forall x\varphi x \supset p)$ and $\forall x(\varphi x \supset p)$. In the Łukasiewicz notation you would be able to see whether what was written was $\Sigma xC\varphi xp$ or $C\Sigma x\varphi xp$.

[3]This makes Aristotle's *M* look like Bocheński's *Z*, which seems to be a kind of contingency operator — strictly a 'contingently true' operator. *Zp* can be defined as *KpMNp*, where *M* is the standard possibility operator, so if *Z* is problematic so is *M*. For this reason the argument which follows can be seen as an argument against all standard modal logic.

which exercised Prior in [13] and [14] was an argument used by Łukasiewicz, though perhaps deriving from Leśniewski, which claimed to shew that if any proposition p and its negation are both possible, i.e. if both Mp and $M{\sim}p$ are true, then every proposition is possible. I have already noted that Łukasiewicz was not sympathetic to modal logic, and his argument that $Mp \wedge M{\sim}p$ implies Mq is referred to by Bocheński on p. 99, in a footnote which presents the proof in a format which is in that same style as Prior subsequently used.

The proof relies on the principle

$$CK\delta p\delta Np\delta q$$

where δ is a variable for all functors. If you substitute M for δ you get

$$CKMpMNpMq$$

which appears to contradict Aristotle, and indeed was taken to do so by both Bocheński and Łukasiewicz, who thought Aristotle had made a mistake. Prior was concerned to stress that the principle in question holds only for truth functions, which are sensitive to nothing more about a proposition than its truth value, and in that case something which holds for both p and not p holds for every q. Bocheński seems to think that this is a serious problem.[4]

Perhaps Łukasiewicz was thinking in some such way as this. If you think of the meaning of the operators of classical propositional logic, their meanings can be specified by truth tables. So that if the only propositions are 1 and 0 then the meaning of each operator is unique. But then, if there can be distinct propositions p and p' which have the same truth value then it would seem that ${\sim}p$ and ${\sim}p'$, say would also have to be distinct propositions, but in that case it would seem that ~ would have to be sensitive to *more* than the truth values of p and p', and this ‘more’ would seem something which a truth table could not explain; in other words ~ would be a function from *propositions*, whatever they are, to propositions. But in that case it would seem that ~ would have to be ambiguous. For suppose that ~* is an operator which denotes the function that is just like the function denoted by ~ except that ${\sim}^*p = {\sim}p'$ and ${\sim}^*p' = {\sim}p$, where p and p' are distinct propositions with the same truth value. This has the consequence that *both* ~ and ~* will satisfy the truth table for negation, so that that table no longer specifies a unique operator. And notice that nothing has been said here about modality. The answer to this problem in fact lies in Saul Kripke’s extension of Prior’s tense logic

[4]Bocheński refers to a 1930 paper. It is not clear from this whether in 1930 Łukasiewicz was using M as a possibility operator. In Bocheński’s footnote the derivation has as premises
1. ΣpMp
2. $EMpMNp$
3. $CK\varphi p\varphi Np\varphi q$
On p. 100 he links 3 to a principle stated using φ as a variable for a propositional operator. It may be that Łukasiewicz’s aim was to criticise Aristotle’s acceptance of two-valued logic, since he may have thought that modal logic is incompatible with the view that every proposition is either ‘the true’ or ‘the false’, and therefore that there are only two of them. The argument was later published in English [11], and is referred to by Prior. George Hughes told me that when he asked Arthur just why he thought that Łukasiewicz might have thought it a good argument Prior replied “old age I should think”.

to include possible worlds, and other such indices, since the truth tables are now relativised to worlds or times. All that of course was not available to Prior or Łukasiewicz, though one can see Prior feeling towards it in [13].[5]

In [15] Prior recommends Robert Feys's [6] as a key source for modal logic. Feys extended propositional logic with the addition of the following:

Possibility	It is possible that α	$M\alpha$	$(\Diamond\alpha)$
Necessity	It is necessary that α	$L\alpha$	$(\Box\alpha)$[6]

In [6], and indeed in [5], Feys was familiar with Lewis's modal logic, and we know also that J.N. Findlay, Prior's teacher at the University of Otago, had access to a copy of [7] when Prior was a student; though there seems little evidence that Prior had a serious interest in 'symbolic' logic until some time in the early 50s or perhaps the late 40s. We do know however that Prior had a strong interest in the history of logic, and in his earlier years, an even stronger interest in Christian theology especially in the issue of freewill and determinism.[7] It is perhaps this interest which led him to develop his tense logic in which he added to modal logic the following tense operators:

It will (at some time) be the case that α	$F\alpha$
It will (always) be the case that α	$G\alpha$
It was once the case that α	$P\alpha$
It has (always) been the case that α	$H\alpha$

That however came later. What I am interested in is what motivated him to adopt the Łukasiewicz notation and to become familiar with Feys's work. We know that Prior knew French and Latin[8], and that he prescribed Bocheński's [3] in his logic classes, and Bocheński's influence was clearly an important factor leading to his espousal of Łukasiewicz's symbolism.[9] In two of his earliest articles ([13] and [14]) Prior cites [2].

[5] While this approach solves the problem of the ambiguity of truth functions in non-extensional contexts it does not, as discussed on pp. 72−75 of [4], where the problem of the ambiguity of negation is discussed, seem applicable to contexts such as belief which do not respect logical equivalence. Such contexts would seem to require a rather different approach.

[6] It is perhaps significant that L was used in this way neither by Bocheński nor Łukasiewicz. Its first recorded use seems to be in [6]. The symbol $\Diamond$ for possibility was used in [7], and the symbol $\Box$ for necessity was devised by F.B. Fitch in 1944 or 1945, (because, Fitch says: "just as possibility is appropriately represented by a symbol balancing precariously on a point, so necessity is appropriately represented by a symbol in a completely stable position." (letter of 11 July 1966.) I shall therefore refer to the $\Diamond/\Box$ notation as the Lewis/Fitch notation. $\Box$ was first used in print in [1]. Bocheński refers to [7] on p. 46n, but only in the context of a remark made by Langford about classical vs contemporary logic ([7] p. 286) and not in the context of Lewis's modal logic. Feys in 1950 was another of the few non-Polish logicians to adopt the Łukasiewicz notation. (The notation of [5] is idiosyncratic.)

[7] Łukasiewicz, like Prior, also appears to think that assigning a determinate truth value (true or false) to all statements is incompatible with indeterminism, and may be another respect in which his influence on Prior appears. (On this see footnote 4.)

[8] Prior was dux of Wairarapa High School in 1931, and even in the 1950s the custom in New Zealand high schools was to teach French and Latin to students in the more academic streams. (While other foreign languages were sometimes taught at some secondary schools they were always in addition to these two.)

[9] It may well have been [3] which got Prior interested in Polish notation. One of his former students told me that he was looking for a suitable logic text, and explained that it would be better to use a good introduction in French than a bad introduction in English. Eventually of course he wrote [15]. More work

3 The syllogistic and negation

What I shall now try to do is suggest another connection between Prior and Bocheński's 1947 book while he was still in the process of becoming a (symbolic) logician.[10] Unlike a presentation of the propositional calculus in which the formulae are truth functions of propositional letters, often *p*, *q*, *r*, ... etc., the atomic formulae of Bocheński's logic are the four kinds of syllogistic propositions. Bocheński uses the letters *a*, *b*, *c*, ... etc. as variables for the terms of a syllogistic proposition, and the letters *U*, *I*, *Y*, and *O* for the four kinds of syllogistic proposition. Thus we have

Uab	Every *a* is a *b*
Iab	Some *a* is a *b*
Yab	No *a* is a *b*
Oab	Some *a* is not a *b*[11]

The traditional syllogistic contains no propositional operators, but it is not difficult to express it in Łukasiewicz's symbolism. Thus the first-figure syllogism known as Barbara can be represented as:

$$CKUabUbcUac.$$

a, *b* and *c* of course are term variables, not propositional variables, and so they cannot be negated — *Na* (~*a*) is not well-formed — but if we want to say that every *a* is a non-*b* we can simply use *Yab* which says that no *a* is a *b*. Since, in Aristotelian terminology a *Y* proposition *converts* we can also use *Yab* to mean that no *b* is an *a*. It is worth noticing here how the way negation is handled in the syllogistic differs from the way it is handled in modern predicate logic. In both Russell's and Łukasiewicz's symbolism you would include individual variables, and use predicate letters, say φ and ψ, in place of *a* and *b* in the following way:

Uab	$\Pi x C\varphi x\psi x$	$\forall x(\varphi x \supset \psi x)$
Iab	$\Sigma x K\varphi x\psi x$	$\exists x(\varphi x \wedge \psi x)$
Yab	$\Pi x C\varphi x N\psi x$	$\forall x(\varphi x \supset {\sim}\psi x)$
Oab	$\Sigma x K\varphi x N\psi x$	$\exists x(\varphi x \wedge {\sim}\psi x)$

still needs to be done on Prior's unpublished letters, but I do know from Jack Copeland that there is a letter 'where Prior waxes on about the merits of Polish notation.', and more relevantly in reference to 'another letter in which Arthur talks about what he was reading in his early years at Canty [Canterbury University College, now the University of Canterbury], while alone and trying to educate himself in formal logic. He mentions B's *Précis* early on, saying he encountered it after getting stuck in to vol 1 of the *Principia*. He says it was his encounter with the Precis that started him writing to B.'

[10] I use the word 'symbolic' here because Prior did seem to regard himself as a 'logician' before he became what we would call a logician. Prior's [12] is called *Logic and the Basis of Ethics*, and he speaks in that book about 'the logician'. He mentions Russell and Whitehead in several places, but never in contexts in which his remarks refer to any particular passage.

[11] In place of *U*, and *Y* the letters *A* and *E* are usually used, but these letters are already used by Łukasiewicz and Bocheński for the propositional operators for disjunction and equivalence. For terms the letters *S* and *P* (subject and predicate) are often used, but Bocheński has used *S* for the necessity operator. It is assumed throughout the syllogistic that no terms are empty.

What modern symbolic logic does in the cases of the expressions just presented is adopt quantifiers which bind individual variables, so that you have atomic formulae of the form φx which can be negated to form conjunctions or implications using the propositional truth functional operators. Thus for instance (i) $\sim\forall x(\varphi x \supset \psi x)$ can be equivalently expressed as $\exists x(\varphi x \wedge \sim\psi x)$ — that is to say, an *O* proposition is the negation of a *U* proposition; while (ii) $\forall x(\varphi x \supset \sim \psi x)$ is already a *Y* proposition. In the syllogistic many of the problems about ambiguities in negative statements which are treated by (i) and (ii) in terms of the scope of a sentential negation operator are handled by the use of what is called the 'Square of Opposition' which sets out the relations between *U*, *I*, *Y* and *O* propositions, by pointing out facts such as that *U* and *O* propositions, and likewise *Y* and *I* propositions are *contradictories* (exactly one will be true), that *U* and *Y* propositions are *contraries* (at most one can be true), that *I* and *O* propositions are *subcontraries* (at least one will be true, and *I* and *O* propositions are respectively *subalterns* of *U* and *Y* propositions, meaning that *Uab* entails *Iab* and *Yab* entails *Oab*.

4 Modality and scope

The syllogistic is undoubtedly more limited than current predicate logic — the most serious respect being that it cannot cope with relational statements which involve formulae like φxy which use n-place predicates for n > 1; but still, as we have seen, in the case of the scope of negation, the traditional method enables us to achieve quite a bit. However, the situation becomes trickier when we come to modality, because of the, to some, contentious, distinction between *de re* and *de dicto* modality. If we say that

(1) All qualified opticians have necessarily passed the opticians' qualifying exam

that may be true when it is represented by the formula

(2) $LC\Pi x\varphi x\psi x$

in the sense that where φ means "is an optician", and ψ means "has passed the opticians' qualifying exam" — i.e., (1) may be true in the *de dicto* sense — yet it does not follow that any optician passed the exam by necessity — that is, (1) is not true in the *de re* sense — since any particular optician might have failed it. Of course they would not, in that case, have been an optician — that is what (2) says. But there is no necessity for anyone to have been an optician. I had planned to be a lawyer, and went in to philosophy by accident because I fell in love with syllogistic logic. What this means is that while (2) is true the formula

(3) $C\Pi x\varphi xL\psi x$

is not. (3) represents the *de re* sense.[12] Real live examples of sentences true in the *de*

[12] In modal predicate logic based on the Russell/Lewis/Fitch notation the distinction is between $\Box\forall x(\varphi x \supset \psi x)$ and $\forall x(\varphi x \supset \Box\psi x)$.

re sense while false in the *de dicto* sense are harder to come by, and examples may sound somewhat artificial. Here is one. We have a Bernese Mountain Dog that we call 'Mrs Geach' (for reasons which philosophers will understand). Now assume that being human is essential to all humans and consider the sentence:

(4) Every living creature called 'Mrs Geach' is necessarily human.

This sentence happens to be false because there is in fact a dog called 'Mrs Geach'; but there is no necessity about that, and (4) could easily have been true. Suppose it were. Then, where φ means "is a living creature called 'Mrs Geach' " and ψ means "is human", (3) would have been true and (2) would have been false.

While the *de re*/*de dicto* distinction can be clearly made in the case of modern modal predicate logic, it is harder to see how the distinction could be expressed in a Bocheński-style syllogistic. Nor is it at all clear just what Aristotle himself thought. One solution may be that Aristotle thought that *all* propositions, at least in the modal syllogistic, had to use terms which, if they applied at all, applied necessarily, which is to say that modal predications are interpreted *de re*, in the sense of (3)[13]. On the other hand it seems that Theophrastus may have argued that all modal statements in the syllogistic have to be understood in a *de dicto* way, as in (2).[14] In either case, though in different ways, there would seem to be no question of ambiguity, and if so maybe the distinction is less important than it might have seemed. And if *this* is so then Bocheński's formulation of the syllogistic will be adequate.

5 Conclusion

What I have tried to do in this paper is tie together two features of Łukasiewicz's work which appear to have made a deep impression on A.N. Prior at a time when he was becoming a 'symbolic' logician and was developing an interest in modal and tense logic. One is Łukasiewicz's logical notation, and the other is Prior's reaction to Łukasiewicz's criticism of Aristotle's concept of possibility. They are features which are made known in French in [2] and in English in [11] and are a crucial part of what made Prior famous, and provides an insight into this development.

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[1] Barcan, (Marcus) R.C. (1946). A functional calculus of first order based on strict implication. *The Journal of Symbolic Logic*, 11, pp. 1–16.

[13] See [17].

[14] Such a reading seems likely, given that Theophrastus appeared to argue that modally necessary universal negative propositions convert. Which is to say that if it is necessary that *Yab* then it is necessary that *Yba*. This holds *de dicto*, since $\Box\forall x(\varphi x \supset \sim \psi x) \equiv \Box\forall x(\psi x \supset \sim \varphi x)$ is a logical law, but does not hold *de re*, since $\forall x(\varphi x \supset \Box \sim \psi x) \equiv \forall x(\psi x \supset \Box \sim \varphi x)$ is not. I gather from Rini that, in contrast to Aristotle, Theophrastus argued that all modalities are de dicto. If this means that no modal operator occurs inside the scope of U, I, Y or O, then all the modalities could be represented by propositional modal operators whose arguments are always U, I, Y or O propositions, and this is what we find in [2].

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