FOIL with constant domains revisited

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Abstract

FOIL is a family of two-sorted first-order modal logics containing both object and intensional variables. Intensional variables are represented by partial functions from worlds to objects and the abstraction operator λ is used to talk about the object (if any) denoted by an intension in a given world.

This paper answers a problem left open in Fitting's [4] by showing that Fitting's axiomatization of FOIL augmented with infinitely many inductively defined rules, CD(k), $k \geq 0$, allows for the construction of a canonical model that is essentially a constant domains model. Moreover, it is shown that the rules CD(k) are derivable in logics where the symmetry axiom B holds. Hence, Fitting's axiomatisation of FOIL is already complete when the underlying logic imposes symmetric models.

1 Introduction

First-order intensional logics (FOIL) are a family of quantified modal logics studied by Melvin Fitting [2, 3, 4] where not only object variables but also intensional variables are present. Object variables are rigid terms that directly represent objects of the model, as terms do in standard quantified modal logics. Intensional variables are non-rigid terms that represent individual concepts – i.e., partial functions from worlds to objects. A complete axiomatisation for some FOIL systems is given in [4].

As Fitting himself notices, his completeness proof for the quantifier-free language gives rise to a canonical model with increasing domains and, hence, it does not work for logics, requiring a constant domains semantics – e.g., it does not work for logics where the symmetry axiom B holds. Thus, Fitting [4, p. 16] presents the following open problem (noted also in [11, p. 571]):

> the structure of the model constructed during the completeness argument is essentially varying domain, while symmetry forces constant domain on us. Without quantifiers available, I don't know how to state something like the Barcan formula, and so I don't know how to reconcile symmetry requirements with the model construction given. Completeness of the $\{\lambda, =\}$ part of FOIL, with an underlying logic of, say, S5 is an interesting question.

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The problem, roughly, is that Fitting [4] constructs a Lindenbaum-Henkin-style model with increasing domains since whenever he has to build a successor world he extends the language with a new set of witnesses. For FOIL based on propositional modal logics such as K, D, T, K4, and S4 – i.e., for logics based on tree-like frames – he is able to prove completeness with respect to a constant domains semantics because

> no quantifiers are present, so there is no way we can "see" the difference between constant and varying domains. [4, p. 16]

Nevertheless this trick does not work for FOIL involving at least one of the axioms B or 5. These axioms correspond to the non-tree-like properties of symmetry and Euclideaness. Hence, in building a canonical model for logics including them we must be able to keep the same language during the entire construction.

This paper provides a positive solution to this problem by showing that:

- FOIL augmented with infinitely many inductively defined rules, CD(k), $k \geq 0$, allows for the construction of a canonical model whose structure during the completeness argument is essentially a constant domains model;[1]
- The rules CD(k) are derivable, for all $k \geq 0$, in logics where the symmetry axiom B holds, and therefore Fitting's axiomatisation of FOIL is complete also when the underlying logic imposes symmetric models.

Moreover, this paper gives a simple solution to a problem noted in [5] for the canonical model construction given in [4]: namely that it is possible for two distinct intensional variables f and g to be interpreted on the same intension without thereby being intersubstitutable *salva veritate* – i.e., it can be that $P(\ldots f \ldots) \in w$ and $P(\ldots g \ldots) \notin w$ for some world w of the canonical model even if f and g denote the same intension. Fitting [5] solves this problem in a way that he himself finds rather artificial: by adding additional worlds (so-called *disambiguation worlds*) to the canonical model in such a way that two intensional variables may be interpreted on the same intension only if syntactically they are the same variable. In this paper the same result is obtained, not by adding disambiguation worlds, but by defining the codomain of intensions as pairs ⟨object, label⟩ in such a way that syntactically distinct intensional variables are interpreted on distinct intensions.

2 Objects, intensions, and constant domains

A crucial question for quantified modal logics is whether terms should behave as rigid designators – i.e., they are bound to denote the same object in all (accessible) worlds of a given model – or as non-rigid designators – i.e., they can denote different objects in different worlds of one and the same model. FOIL are two-sorted quantified modal logics encompassing terms of both kinds. Object variables denote objects directly and irrespectively of the particular world of the model under consideration. Intensional

[1] What we present here has similarities with the rules R6 and R7 introduced by Thomason [9] and with the rule EBR in Corsi [1].

variables, instead, denote functions from worlds to objects; moreover these functions need not be total, hence it might well be that an intension does not denote any object in some world. To exemplify the distinction, an object variable behaves like the proper name ‘Donald J. Trump’ and an intensional variable behaves like the expression ‘the president of the US’. Whereas the former denotes uniformly a particular object, the latter denotes different objects in different worlds: it denotes Donald J. Trump in the actual world, but in a reasonable alternative it denotes Hillary Clinton and in a more remote alternative it simply lacks a denotation (say in a world where the Confederacy had won the American Civil War).

Another important distinction for formulas involving intensional terms is whether they are used to speak about the intension as such, as in ‘the president of the US is more influent than the former protagonist of the TV show The Apprentice’, or to speak about the object actually picked out by that intension, as in ‘the president of the US is one of the richest men of the world’: in the first case the intensional term ‘the president of the US’ is used to talk about the presidential role irrespectively of who is the actual president, whereas in the latter it is used to talk about the individual who actually is the president – i.e., Donald J. Trump.

Following [8, 10], Fitting uses the predicate abstraction operator λ as a scoping mechanism for designation by intensions: the formula $\langle \lambda x.P(x)\rangle f$ says that the object denoted by the intension f satisfies the predicate $\langle \lambda x.P(x)\rangle$. Since intensions are partial functions from worlds to objects, if f does not denote in w then the value of $\langle \lambda x.\neg A\rangle f$ differs from that of $\neg\langle \lambda x.A\rangle f$: the first formula will be false and the second true at w since no formula can be true of the denotation of f in worlds where f does not denote. In the case of modalities a scoping mechanism is needed because the object denoted by an intension changes from world to world. Hence, in a world w, the value of $\langle \lambda x.\Box A\rangle f$ might differ from that of $\Box\langle \lambda x.A\rangle f$. For the first formula, which can be read as ‘f has the property of being necessarily A’, we have first to determine the object o denoted by f in w and then to move to accessible worlds to see whether o satisfies $A(x)$ therein. For the second formula, which can be read as ‘it is necessary that f has property A’, we have first to move to accessible worlds and then to see whether the object therein denoted by f satisfies $A(x)$. To sum up, we need λ as a scoping mechanism because intensions might fail to denote and, when they denote, they do it non-rigidly.

As to be expected when constructing a world w of a canonical model witnesses are necessary for abstraction formulas $\langle \lambda x A\rangle f$, i.e. if $\langle \lambda x.A\rangle f \in w$ then there must be a parameter p such that $\langle \lambda x.x = p\rangle f \in w$ and $A[p/x] \in w$. When moving from a world w to a related world v, in order to have witnesses in v for λ-abstraction formulas a typical strategy is that of enlarging the language of w so as to have fresh witnesses. But by doing so, one gets models with increasing domains. A parallel situation takes place when quantifiers are present and witnesses are needed for existential formulas: $\exists x A(x)$. As well known, see [9], in this case the Barcan formula comes to rescue and witnesses in w can be used again as witnesses in worlds accessible from w, therefore Kripke models with constant domains can be constructed.

In [4] Fitting considers FOIL based on a quantifier-free language with λ-abstraction operator and identity and he provides completeness with respect to Kripke models with constant domains. The strategy used is a variation of the usual Henkin-style construction which gives rise to worlds with expanding domains, but then the final canonical model is defined as to have the same domain for each world – i.e., the union of the domains provided by the Henkin-style construction. This is feasible because the quantifier-free language cannot distinguish between increasing and constant domains: each logic that is complete with respect to a given class of models with increasing domains is complete also with respect to the corresponding class of models with constant domains. In particular, completeness (w.r.t. both increasing and constant domains) can be proved for the tree-like FOIL based on propositional modal logics K, D, T, K4, and S4. What cannot be proved using an expanding set of witnesses is the completeness of non-tree-like FOIL such as those involving the symmetry axiom B or the Euclidean axiom 5: these logics are not complete with respect to models with increasing domains, but only with respect to constant domains ones.

For the FOIL systems without quantifiers, infinitely many rules are needed to provide a Henkin-style construction involving a constant set of witnesses. Here are the rules, for all $k \geq 0$,

$$\frac{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f)\dots)}{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg \mathsf{D}f)\dots)} \text{ CD(k), } y \text{ not free in } A_i$$

where $\mathsf{D}f$ (to be read as 'f denotes') is an abbreviation for $\langle \lambda x.\top \rangle f$. If $k = 0$, we get

$$\frac{A_0 \wedge \mathsf{D}f \to \langle \lambda x.x \neq y \rangle f}{A_0 \to \neg \mathsf{D}f} \text{ CD(0), } y \text{ not free in } A_0$$

and its meaning is clear: if at the world under consideration f is defined and A_0 is true, there must be an element of the domain of that world which is the denotation of f in that world. Analogously, rule CD(k), with k arbitrary, says that if in a world that is k-accessible (i.e., a world that is accessible in k steps) from the actual one the intension f is defined and A_k is true, then there must be an element of the domain that is the denotation of f in the k-accessible one.

What is most remarkable is that in the presence of axiom $B : A \to \Box\Diamond A$ the rule CD(k) is derivable from CD(0) for each $k > 0$, see Lemma 6. Moreover, CD(0) is a theorem of FOIL, see lemma 5.[2] This implies that all FOIL systems with B as a theorem can be proved complete with respect to models with constant domains.

3 Syntax

We consider a signature containing, for each pair $\langle n, m \rangle$ with $n, m \in \mathbb{N}$, a set, at most denumerable, of $n + m$-ary logical relational symbols, denoted by $P^{n,m}, R^{n,m}$.[3] There are no individual constants nor function symbols, but they can be added without any difficulty. The two sorted language contains a denumerable set of *object variables*, OBJ, to be denoted by $x, y, z, \dots$, and a denumerable set of *intensional variables*, INT, denoted by $f, g, h, \dots$ The language contains also the $2 + 0$-ary identity symbol $=$ and the logical symbols $\bot, \to$ $\Box, \lambda$. An *atomic formula* is any expression $P^{n,m}(\vec{x}, \vec{f})$ – where

[2] Let us recall, *en passant*, that the Barcan formula is derivable in classically quantified logics when B is a theorem.

[3] For simplicity's sake and without loss of generality we let the object variables precede the intensional variables.

P is a $n+m$-ary relational symbol, $\vec{x}$ is an n-ary list of object variables, and $\vec{f}$ is an m-ary list of intensional variables – or an *identity atom* $x = y$. Whenever possible we will omit the superscripts from the relational symbols. The (quantifier-free) language $\mathcal{L}$ is generated by the following grammar:[4]

$$A \ ::= \ P^{n,m}(\vec{x}, \vec{f}) \,|\, x = y \,|\, \bot \,|\, A \to A \,|\, \Box A \,|\, \langle \lambda x.A \rangle f \qquad (\mathcal{L})$$

The symbols $\top, \neg, \wedge, \vee, \leftrightarrow, \Diamond$ and $\neq$ are defined as usual, and the formula:

$$\mathsf{D}f \text{ (to be read as `} f \text{ denotes') abbreviates } \langle \lambda x.\top \rangle f \qquad \text{(def. D)}$$

We use $A, B, C, \ldots$ for arbitrary formulas and α, β for both object and intensional variables. Moreover, we abbreviate the formula $\langle \lambda x.\langle \lambda y.A \rangle g \rangle f$ as $\langle \lambda x, y.A \rangle g.f$. λ is the only variable-binding operator of $\mathcal{L}$. Free and bound occurrences of object variables in a formula are defined in the standard way – e.g., in $\langle \lambda x.A \rangle f$ all occurrences of x are bound by λx. Intensional variables are always free.

The symbol $\equiv$ denotes syntactic identity. Without loss of generality, we assume that the variables occurring free in a formula are different from the bound ones, and we identify formulas that differ only in the name of bound variables. By $A[y/x]$ we denote the formula that is obtained by substituting each free occurrence of x in A with an occurrence of y, provided that y is free for x in A – that is to say, no free occurrence of x becomes a bound occurrence of y after the substitution is performed. The formula $A[g/f]$ is defined analogously. Having identified formulas differing only in the name of bound variables, we can assume that y is always free for x in $A[y/x]$.

4 Semantics

A *constant domains model* is a tuple $\mathcal{M} = \langle W, R, D_O, D_L, D_I, V \rangle$ where:

1. $\langle W, R \rangle$ is a frame;
2. D_O is a non-empty set of *objects*;
3. D_L is a non-empty set of *labels* $\ell_{\widehat{f}}, \ell_{\widehat{g}} \ldots$;
4. D_I is a set of *intensions* such that, for each $\ell_{\widehat{f}} \in D_L$, D_I contains a partial function $\widehat{f} : W \longrightarrow D_O \times D_L$ such that:
   - if $\widehat{f}$ is defined for $w \in W$, then $\widehat{f}(w) = \langle o, \ell_{\widehat{f}} \rangle$, for some $o \in D_O$,
   - if, instead, $\widehat{f}$ is not defined for $w \in W$, then $\widehat{f}(w) = \langle \ell_{\widehat{f}}, \ell_{\widehat{f}} \rangle$;
5. V is a *valuation function* such that:
   - $V(P^{n,m}, w) \subseteq (D_O)^n \times (D_I)^m$, and
   - $V(=, w) = \{\langle o, o \rangle : o \in D_O\}$.

[4]In Fitting's [4] terminology this is the language $\mathcal{L}^{\{\lambda,=\}}$.

$\mathcal{M}$ so defined is said to be a *constant domains model* because the domain of interpretation of the relational symbols and the variables doesn't vary depending on the world under consideration, it is always D_O.

We say that $\mathcal{M}$ has *complete intensions* (*CI-model*) if each $\widehat{f} \in D_I$ is a total function.

An *assignment* of a model $\mathcal{M}$ is a function σ mapping each individual variable to a member of D_O and each intensional variable to a member of D_I.

We say that an intentional variable f is *defined for a world w* under the assignment σ iff $\sigma(f)(w) \in (D_O \times \{\ell_{\widehat{f}}\})$.

When f is defined for w under the assignment σ, we define the function $\sigma^\star$ to the effect that

$$\sigma^\star(f)(w) = o \qquad \text{iff} \qquad \sigma(f)(w) = \langle o, \ell_{\widehat{f}} \rangle$$

We use $\sigma^{x \triangleright o}$ ($\sigma^{f \triangleright i}$) for the assignment behaving like σ except for the variable x (f) that is mapped to the object $o \in D_O$ (the intension $i \in D_I$, respectively).

The notion of *satisfaction* of a formula A in a world w of a model $\mathcal{M}$ under an assignment σ, to be denoted by $\sigma \models^{\mathcal{M}}_w A$ (possibly omitting the $\mathcal{M}$), is defined as follows.

Definition 1. (Satisfaction)

$\sigma \models_w P^{n,m}(\vec{x}, \vec{f})$	*iff*	$\langle \sigma(x_1), \ldots, \sigma(x_n), \sigma(f_1), \ldots, \sigma(f_m) \rangle \in V(P^{n,m}, w)$
$\sigma \models_w x = y$	*iff*	$\sigma(x) = \sigma(y)$
$\sigma \not\models_w \bot$		
$\sigma \models_w A \to B$	*iff*	$\sigma \models_w A$ *implies* $\sigma \models_w B$
$\sigma \models_w \Box A$	*iff*	*for each* $v \in W$, wRv *implies* $\sigma \models_v A$
$\sigma \models_w \langle \lambda x.A \rangle f$	*iff*	*there is an* $o \in D_O$ *s.t.:* $\sigma^\star(f)(w) = o$ *and* $\sigma^{x \triangleright o} \models_w A$

Note that

$\sigma \models_w \langle \lambda x.\top \rangle f$ iff there is an $o \in D_O[\sigma^\star(f)(w) = o \wedge \sigma^{x \triangleright o} \models_w \top]$ iff there is an $o \in D_O[\sigma^\star(f)(w) = o]$;

$\sigma \models_w \langle \lambda x.(x = p) \rangle f$ iff there is an $o \in D_O[\sigma^\star(f)(w) = o \wedge \sigma^{x \triangleright o} \models_w x = p]$ iff there is an $o \in D_O[\sigma^\star(f)(w) = o \wedge o = \sigma(p)]$ iff $[\sigma^\star(f)(w) = \sigma(p)]$;

If there is no $o \in D_O$ such that $[\sigma^\star(f)(w) = o]$ then not $(\sigma \models_w \langle \lambda x.A \rangle f)$ and so $\sigma \not\models_w \langle \lambda x.A \rangle f$.

A formula A is *true at a world w of a model* $\mathcal{M}$, $\models^{\mathcal{M}}_w A$, iff for all assignments σ, $\sigma \models^{\mathcal{M}}_w A$.

A formula A is *true on a model* $\mathcal{M}$, $\models^{\mathcal{M}} A$, iff for all w, $\models^{\mathcal{M}}_w A$.

A formula A is *valid on a class of models* C, $C \models A$, iff for all models $\mathcal{M}$ in C, $\models^{\mathcal{M}} A$.

A *logic* L is the set of all formulas that are valid on a given class of models. A model is a *model for a logic* L if it makes true all formulas of L.

5 Axiomatic systems

Definition 2 (FOIL)**.** *The axiomatisation of the set of $\mathcal{L}$-formulas that are valid on the class of all models is given by the following axioms and rules [4]:*

- *Axioms:*

1. *All $\mathcal{L}$-instances of propositional tautologies*
2. $\Box(A \to B) \to (\Box A \to \Box B)$
3. $\langle \lambda x.A \to B \rangle f \to (\langle \lambda x.A \rangle f \to \langle \lambda x.B \rangle f)$
4. $\langle \lambda x.A \rangle f \to A$, *for x not free in A*
5. $\langle \lambda x.A \rangle f \to \langle \lambda y.A[y/x] \rangle f$, *for y free for x in A*
6. $\mathsf{D}f \to (\langle \lambda x.A \rangle f \vee \langle \lambda x.\neg A \rangle f)$
7. $x = x$
8. $x = y \to (P[x/z] \to P[y/z])$, *for P an atomic formula;*
9. $x = y \to \Box(x = y)$
10. $x \neq y \to \Box(x \neq y)$
11. $\mathsf{D}f \to \langle \lambda y, x.x = y \rangle f.f$

- *Rules:*

$$\frac{A \quad A \to B}{B}\ MP \qquad\qquad \frac{A}{\Box A}\ N \qquad\qquad \frac{A \to B}{\langle \lambda x.A \rangle f \to \langle \lambda x.B \rangle f}\ \lambda\text{-reg}$$

Lemma 3. *The following $\mathcal{L}$-formulas are theorems of* FOIL*:*

1. $\vdash_{\mathsf{FOIL}} \mathsf{D}f \to (\neg\langle \lambda x.A \rangle f \leftrightarrow \langle \lambda x.\neg A \rangle f)$
2. $\vdash_{\mathsf{FOIL}} \langle \lambda x.A \rangle f \leftrightarrow (\mathsf{D}f \wedge A)$ provided x not free in A
3. $\vdash_{\mathsf{FOIL}} (\langle \lambda y.x = y \rangle f \wedge \langle \lambda y.z = y \rangle f) \to (x = z)$
4. $\vdash_{\mathsf{FOIL}} x = y \to (A[x/z] \to A[y/z])$
5. $\vdash_{\mathsf{FOIL}} \langle \lambda x.x = y \rangle f \to (\langle \lambda x.A \rangle f \leftrightarrow A[y/x])$

Proof. For items (1), (2), and (3) see [4, Proposition 4.1]. (4) follows from axioms (8) and (9). For (5) we have:

(a)	$\vdash x = y \to (A[x/x] \leftrightarrow A[y/x])$	Lemma 3.4
(b)	$\vdash \langle \lambda x.x = y \rangle f \to (\langle \lambda x.A \rangle f \leftrightarrow \langle \lambda x.A[y/x] \rangle f)$	from (a) by λ-reg
(c)	$\vdash \langle \lambda x.x = y \rangle f \to (\langle \lambda x.A \rangle f \to A[y/x])$	from (b) by axiom 4
(d)	$\vdash Df \wedge A[y/x] \to \langle \lambda x.A \rangle f$	Lemma 3.2
(e)	$\vdash Df \to (A[y/x] \to \langle \lambda x.A \rangle f)$	from (d)
(f)	$\vdash \langle \lambda x.x = y \rangle f \to (A[y/x] \to \langle \lambda x.A \rangle f)$	from (e) by Lemma 3.2
(g)	$\vdash \langle \lambda x.x = y \rangle f \to (A[y/x] \leftrightarrow \langle \lambda x.A \rangle f)$	from (c) and (f)

□

Definition 4. *The calculus* FOIL.CD *is given by adding to* FOIL *the CD-rule:*

$$\frac{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f)\dots)}{A_0 \to \Box(A_1 \to \cdots \to \Box(A_k \to \neg\mathsf{D}f)\dots)} \text{ CD(k)}, k \geq 0, y \text{ not free in } A_i \quad \text{(CD-rule)}$$

Lemma 5. *The rule CD*(0) *is derivable in* FOIL*:*

$$\frac{(A_o \wedge \mathsf{D}f) \to \langle\lambda x.x \neq y\rangle f}{A_0 \to \neg\mathsf{D}f} \text{ where } y \text{ is not free in } A_0$$

Proof.

(a)	$A_0 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f$	Assumption
(b)	$A_0 \to [\langle\lambda x.\top\rangle f \to \langle\lambda x.x \neq y\rangle f]$	from (a) and def. D
(c)	$\langle\lambda y.A_0\rangle f \to [\langle\lambda y, x.\top\rangle f.f \to \langle\lambda y, x.x \neq y\rangle f.f]$	from (b) by λ-reg
(d)	$A_0 \wedge \mathsf{D}f \to [\langle\lambda y, x.\top\rangle f.f \to \langle\lambda y, x.x \neq y\rangle f.f]$	from (c) by Lemma 3.2
(e)	$A_0 \wedge \mathsf{D}f \to [\mathsf{D}f \wedge \langle\lambda x.\top\rangle f \to \langle\lambda y, x.x \neq y\rangle f.f]$	from (d) by Lemma 3.2
(f)	$A_0 \wedge \mathsf{D}f \to [\mathsf{D}f \wedge \mathsf{D}f \to \langle\lambda y, x.x \neq y\rangle f.f]$	by def. D
(g)	$A_0 \to [\mathsf{D}f \to \langle\lambda y, x.x \neq y\rangle f.f]$	from (f)
(h)	$A_0 \to [\mathsf{D}f \to \langle\lambda y.\neg\langle\lambda x.x = y\rangle f\rangle f]$	from (g) by Lemma 3.1
(i)	$A_0 \to [\mathsf{D}f \to \neg\langle\lambda y, x.x = y\rangle f.f]$	from (h) by Lemma 3.1
(j)	$A_0 \to [\mathsf{D}f \to \langle\lambda y, x.x = y\rangle f.f]$	from (i) by Axiom 11
(k)	$A_0 \to \neg\mathsf{D}f$	from (i) and (j).

□

Now we show a key lemma concerning FOIL augmented with axiom $B : A \to \Box\Diamond A$.

Lemma 6. *The* CD(k) *rules,* $k \geq 0$*, are derivable in* FOIL.B.

Proof. As is well known, the following rules are derivable from axiom B:

$$\frac{\Diamond A \to B}{A \to \Box B}\ DRB \qquad \frac{A \to \Box B}{\Diamond A \to B}\ DRB'$$

We show that CD(2) is derivable in FOIL.B.[5]

(a)	$A_0 \to \Box(A_1 \to \Box(A_2 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f))$	Assumption
(b)	$\Diamond A_0 \to (A_1 \to \Box(A_2 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f))$	from (a) by *DRB′*
(c)	$(\Diamond A_0 \wedge A_1) \to \Box(A_2 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f)$	from (b)
(d)	$\Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f)$	from (c) by *DRB′*
(e)	$(\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to (\mathsf{D}f \to \langle\lambda x.x \neq y\rangle f)$	from (d)
(f)	$\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to \neg\mathsf{D}f$	from (e) by CD(0)
(g)	$\Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \to \neg\mathsf{D}f)$	from (f)
(h)	$(\Diamond A_0 \wedge A_1) \to \Box(A_2 \to \neg\mathsf{D}f)$	from (g) by DRB
(i)	$\Diamond A_0 \to (A_1 \to \Box(A_2 \to \neg\mathsf{D}f))$	from (h)
(j)	$A_0 \to \Box(A_1 \to \Box(A_2 \to \neg\mathsf{D}f))$	from (i) by *DRB*

Analogously, CD(k) is derivable in FOIL.B for all $k \geq 0$. □

(D) $\Box A \to \Diamond A$

(T) $\Box A \to A$

(4) $\Box A \to \Box\Box A$

(5) $\Diamond A \to \Box\Diamond A$

(B) $A \to \Box\Diamond A$

(CI) $\mathsf{D}f$

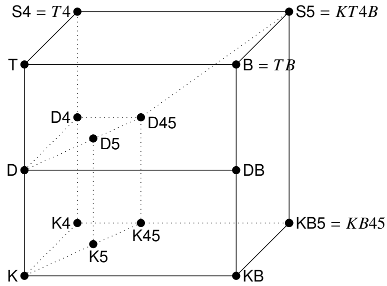

Figure 1: Additional axioms and cube of propositional modal logics

Theorem 7 (Soundness of FOIL.CD.S)**.** *Let* S *be any of the propositional modal logics of figure 1. Each theorem of* FOIL.CD.S *is valid in the class of all frames with constant domains for the propositional modal logic* S.

Proof. We prove only that the rule CD(1) preserves truth at each point of a constant domains model:

$$\frac{A_0 \to \Box(A_1 \wedge Df \to \langle\lambda x.x \neq y\rangle f)}{A_0 \to \Box(A_1 \to \neg Df)} \qquad y \text{ not free in } A_0, A_1$$

Suppose, by *reductio*, that for some w of a constant domains model $\mathcal{M}$:

$$\models_w A_0 \to \Box(A_1 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f) \tag{1}$$

and that for some assignment σ:

$$\sigma \not\models_w A_0 \to \Box(A_1 \to \neg\mathsf{D}f) \tag{2}$$

(2) entails that there is a u such that wRu, $\sigma \models_u A_1$ and $\sigma \models_u \mathsf{D}f$ – i.e., $\sigma(f)(u) = o$ for some $o \in D_O$. From (1) we have that for all assignments τ, $\tau \models_w A_0 \to \Box(A_1 \wedge \mathsf{D}f \to \langle\lambda x.x \neq y\rangle f)$, in particular also for the assignment $\sigma^{y \triangleright o}$. Hence, $\sigma^{y\triangleright o} \models_u \langle\lambda x.x \neq y\rangle f$, and so $\sigma^{y\triangleright o, x\triangleright\sigma^*(f)(u)} \models_u x \neq y$, we have thus reached a contradiction: $o \neq o$. □

6 Completeness

We are ready to present a general strategy to prove completeness theorems for all the FOIL systems based on the propositional modal logics of figure 1 plus the CD-rule. If the propositional logic proves the formula B, we do not need to add the CD-rule, since it is derivable (see Lemma 6).

We prove strong completeness by the usual Henkin-style technique, cf. [1]. Let P be a denumerable set of fresh object variables (to be called *parameters*) and let $\mathcal{L}^P$ be the language obtained by adding the set P to $\mathcal{L}$ and by imposing that parameters cannot be bound by λ. In this section, we use L (L^P) to denote any logic that includes FOIL.CD over the language $\mathcal{L}$ ($\mathcal{L}^P$, respectively). Moreover, we use Δ for a set of formulas of $\mathcal{L}$ or of $\mathcal{L}^P$ (depending on the context).

[5]See [7, p. 295].

Definition 8.

- Δ *is* L^P-consistent *iff* $\Delta \nvdash_{\mathsf{L}^P} \bot$.
- Δ *is* $\mathcal{L}^P$-complete *iff* *for all* $A \in \mathcal{L}^P$, *either* $A \in \Delta$ *or* $\neg A \in \Delta$.
- Δ *is* $\Diamond^k$-P-rich *iff* *if* $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f) \ldots) \in \Delta$ *then* $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x(x = p)\rangle f) \ldots) \in \Delta$ *for some* $p \in P \cup OBJ$.
- Δ *is* $\Box^k$-P-inductive *iff* *if* $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \langle \lambda x.x \neq p\rangle f) \ldots) \in \Delta$ *for all* $p \in P \cup OBJ$ *then* $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \rightarrow \neg \mathsf{D}f) \ldots) \in \Delta$.
- Δ *is* $\mathcal{L}^P$-saturated *iff it is* L^P*-consistent,* $\mathcal{L}^P$*-complete, and* $\Diamond^k$*-P-rich, for all* $k \in \mathbb{N}$.

Remark 9. *Notice that* Δ *is* $\Diamond^0$*-P-rich iff if* $A_0 \wedge \mathsf{D}f \in \Delta$ *then, for some* $p \in P \cup OBJ$, $A_0 \wedge \langle \lambda x.x = p\rangle f \in \Delta$.
Δ *is* $\Box^0$*-P-inductive iff if* $A_0 \wedge \mathsf{D}f \rightarrow \langle \lambda x(x \neq p\rangle f) \in \Delta$ *for all* $p \in P \cup OBJ$ *then* $A_0 \rightarrow \neg \mathsf{D}f \in \Delta$.

Lemma 10. *If* Δ *is* $\mathcal{L}^P$*-saturated, then it is* $\Box^k$*-P-inductive for all* $k \in \mathbb{N}$.

Proof. Let $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \langle \lambda x.x \neq p\rangle f) \ldots) \in \Delta$ for all $p \in P \cup OBJ$ and suppose by *reductio* that $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \rightarrow \neg \mathsf{D}f) \ldots) \notin \Delta$. As Δ is $\mathcal{L}^P$-complete, $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f) \ldots)) \in \Delta$.

Δ is $\Diamond^k$-P-rich, so $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f) \ldots) \in \Delta$, for some $p \in P \cup OBJ$. But this contradicts the L^P-consistency of Δ since by hypothesis, for all $p \in P \cup OBJ$, $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \langle \lambda x.x \neq p\rangle f) \ldots) \in \Delta$ and therefore, by Lemma 3.1, $A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \neg\langle \lambda x.x = p\rangle f) \ldots) \in \Delta$ holds for all $p \in P \cup OBJ$. □

Lemma 11.

1. *If* $\Delta \vdash_{\mathsf{L}^P} A$ *and no parameter of* P *occurs in* $\Delta \cup \{A\}$*, then* $\Delta \vdash_{\mathsf{L}} A$.
2. *Let* $\Delta \vdash_{\mathsf{L}^P} A[p/x]$*, where* p *doesn't occur in* Δ*, and let* z *be a variable not occurring in the derivation of* $A[p/x]$ *from* Δ*. Then* $\Delta \vdash_{\mathsf{L}^P} A[z/x]$.
3. *If* Δ *is* L*-consistent and no* $p \in P$ *occurs in* Δ*, then it is also* L^P*-consistent.*

Proof. To prove (1) and (2), assume that $p_1, \ldots, p_n$ are all the parameters occurring in the derivation $\mathcal{D}$ of $\Delta \vdash_{\mathsf{L}^P} A$ (of $\Delta \vdash_{\mathsf{L}^P} A[p/x]$, respectively), and that $z, z_1, \ldots z_n$ are variables not occurring therein. By replacing each occurrence of $p_{(i)}$ in $\mathcal{D}$ with an occurrence of $z_{(i)}$ we obtain a derivation of $\Delta \vdash_{\mathsf{L}} A$ or of $\Delta \vdash_{\mathsf{L}^P} A[z/x]$, respectively. (3) follows from (1). □

Lemma 12 (Lindenbaum-Henkin)**.** *If* Δ *is an* L*-consistent set of formulas of* $\mathcal{L}$*, then there is an* $\mathcal{L}^P$*-saturated set* $\Delta^\star$*, for some denumerable set of parameters* P*, such that* $\Delta^\star \supseteq \Delta$.

Proof. Let $B_0, B_1, B_2, \ldots, B_n, B_{n+1}, \ldots$ be an enumeration of all the formulas of $\mathcal{L}^P$. Let us define the following chain:

- $\Delta_0 = \Delta$.
- In order to define Δ_{n+1} we consider the set Δ_n and the formula B_n:
  1. If $\Delta_n \cup \{B_n\}$ is not L^P-consistent, let $\Delta_{n+1} = \Delta_n \cup \{\neg B_n\}$.
  2. If $\Delta_n \cup \{B_n\}$ is L^P-consistent, we distinguish two caes:
     (a) If $B_n \equiv A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f)\ldots)$, for some $A_0, \ldots, A_k$, let $\Delta_{n+1} = \Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f)\ldots)\}$ for some $p \in P \cup OBJ$ such that the resulting set is L^P-consistent;
     (b) Else, $\Delta_{n+1} = \Delta_n \cup \{B_n\}$.

Lemma 13. *Each member of the chain thus defined is* L^P*consistent.*

Proof. By induction on n we show that Δ_n is L^P-consistent for every n.

Δ_0 is L-consistent since Δ is so by hypothesis and therefore, by Lemma 11.3, Δ_0 is L^P-consistent.

Let us assume, by induction hypothesis, that Δ_n is L^P-consistent and consider case (2)(a). Let $B_n \equiv A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f)\ldots)$ and suppose, by *reductio*, that $\Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f)\ldots)\}$ is not L^P-consistent for all $p \in P \cup OBJ$.

Then $\Delta \vdash_{\mathsf{L}^P} (G \wedge A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f)\ldots)) \rightarrow \bot$, for all $p \in P \cup OBJ$, where G is the (finite) conjunction of the formulas of $(\Delta_n - \Delta)$.

Hence $\vdash_{\mathsf{L}^P} (C \wedge G \wedge A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p^\star\rangle f)\ldots)) \rightarrow \bot$, where C is a conjunction of formulas of Δ and $p^\star$ is a parameter occurring neither in G nor in B_n. $p^\star$ doesn't occur in C, since no parameter occurs in formulas of Δ.

Thus, by modal reasoning and Lemma 11.2, for some fresh variable z, $\vdash_{\mathsf{L}^P} C \wedge G \wedge A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \neg\langle \lambda x.x = z\rangle f)\ldots)$ and, by Lemma 3.1, $\vdash_{\mathsf{L}^P} C \wedge G \wedge A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \mathsf{D}f \rightarrow \langle \lambda x.x \neq z\rangle f)\ldots)$.

Whence, $\vdash_{\mathsf{L}^P} C \wedge G \wedge A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \neg\mathsf{D}f)\ldots)$, by rule CD(k). Hence, $\Delta_n \vdash_{\mathsf{L}^P} A_0 \rightarrow \Box(A_1 \rightarrow \cdots \rightarrow \Box(A_k \wedge \neg\mathsf{D}f)\ldots)$ contrary to the L^P-consistency of $\Delta_n \cup \{B_n\}$.

By the induction principle, each Δ_n is L^P-consistent. □

Let

$$\Delta^\star = \bigcup_{n \in N} \Delta_n$$

The set $\Delta^\star$ is L^P-consistent since each Δ_n is so. Moreover $\Delta \subseteq \Delta^\star$ and $\Delta^\star$ is $\mathcal{L}^P$-complete and $\Diamond^k$-P-rich by construction. □

Lemma 14 (Diamond-lemma)**.** *If w is an $\mathcal{L}^P$-saturated set of formulas and $\Diamond A \in w$ then there is a set v of $\mathcal{L}^P$-formulas such that:*

1. v is $\mathcal{L}^P$-saturated;

2. *$A \in v$;*

3. *$v \supseteq \Box^-(w)$, where $\Box^-(w) = \{A : \Box A \in w\}$;*

4. *for each $p \in P \cup OBJ$, $[p]_w = [p]_v$, where $[p]_w = \{a : p = a \in w\}$.*

Proof. Let $B_0, B_1, B_2, \dots B_n, B_{n+1} \dots$ be an enumeration of all $\mathcal{L}^P$-formulas. Let us define the following chain:

- $\Delta_0 = \Box^-(w) \cup \{A\}$;
- Given Δ_n and B_n, we define Δ_{n+1}:
  1. If $\Delta_n \cup \{B_n\}$ is not L^P-consistent, let $\Delta_{n+1} = \Delta_n \cup \{\neg B_n\}$;
  2. If $\Delta_n \cup \{B_n\}$ is L^P-consistent, then we distinguish two cases:
     (a) If $B_n \equiv A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_k \wedge \mathsf{D}f)\dots)$, for some $A_0, \dots, A_k$, let $\Delta_{n+1} = \Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f)\dots)\}$ for some $p \in P \cup OBJ$ such that the resulting set is L^P-consistent;
     (b) Else, $\Delta_{n+1} = \Delta_n \cup \{B_n\}$.

Lemma 15. *Each element of the chain is L^P-consistent.*

Proof. By induction on n.

Δ_0 is L^P-consistent by modal reasoning. Assume, by induction hypothesis, that Δ_n is L^P-consistent. It will be enough to consider the case (2)(a).

Suppose by *reductio* that there is no $p \in P \cup OBJ$ such that the set

$\Delta_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f))\}$ is L^P-consistent. Then, for all $p \in P \cup OBJ$, it holds that
$\Delta_n \vdash_{\mathsf{L}^P} (A_0 \wedge \Diamond(A_1 \wedge \dots \wedge \Diamond(A_k \wedge \mathsf{D}f \wedge \langle \lambda x.x = p\rangle f))) \to \bot$, and by modal reasoning, $\Delta_n \vdash_{\mathsf{L}^P} (A_0 \to \Box(A_1 \to \dots \to \Box(A_k \wedge \mathsf{D}f \to \neg\langle \lambda x.x = p\rangle f)))$. By Lemma 3.1, $\Delta_n \vdash_{\mathsf{L}^P} (A_0 \to \Box(A_1 \to \dots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq p\rangle f)))$.

Moreover, Δ_n is just $\Box^-(w) \cup \{C_1, \dots, C_m\}$ for some finite set of formulas $\{C_1, \dots, C_m\}$, therefore, where $C \equiv C_1 \wedge \dots \wedge C_m$, for each $p \in P \cup OBJ$ we have that $\Box^-(w) \vdash_{\mathsf{L}^P} C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq p\rangle f))$.

Thus $w \vdash_{\mathsf{L}^P} \Box(C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq p\rangle f)))$ and trivially $w \vdash_{\mathsf{L}^P} \top \to \Box(C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \wedge \mathsf{D}f \to \langle \lambda x.x \neq p\rangle f)))$ for all $p \in P \cup OBJ$. Since w is $\mathcal{L}^P$-saturated, by Lemma 10, w is $\Box^j$-P-inductive for all $j \in \mathbb{N}$, hence, in particular w is $\Box^{k+1}$-P-inductive, so $w \vdash_{\mathsf{L}^P} \top \to \Box(C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \to \neg\mathsf{D}f)))$. It follows that $w \vdash_{\mathsf{L}^P} \Box(C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \to \neg\mathsf{D}f)))$, which implies that $(C \wedge A_0 \to \Box(A_1 \to \dots \to \Box(A_k \to \neg\mathsf{D}f))) \in \Box^-(w)$, and therefore
$\Delta_n \vdash_{\mathsf{L}^P} A_0 \to \Box(A_1 \to \dots \to \Box(A_k \to \neg\mathsf{D}f))$. But this contradicts the L^P-consistency of $\Delta_n \cup \{B_n\}$. □

Let
$$v = \bigcup_{n \in \mathbb{N}} \Delta_n$$

The set v is L^P-consistent. All the properties of the lemma hold for v: (1)–(3) hold by construction.

As to (4), if $a \in [p]_w$, then $(p = a) \in w$, then by axiom (9), $\Box(p = a) \in w$, $(p = a) \in \Box^-(w)$, $(p = a) \in v$, so $a \in [p]_v$. If, instead $a \notin [p]_w$, then $(p = a) \notin w$, then $(p \neq a) \in w$ since w is $\mathcal{L}^P$–saturated. By axiom (9), $\Box(p \neq a) \in w$, so $(p \neq a) \in \Box^-(w)$, $(p \neq a) \in v$, so $a \notin [p]_v$. □

Remark 16. *The Diamond Lemma 14 is the key step in the construction of a canonical model based on a constant domains construction because it ensures that we can saturate all worlds of a canonical model with respect to a single set of parameters P.*

In the proof of Lemma 14, in order to make sure that the set Δ_{n+1} is both L^P-consistent and $\Diamond^k$-P-rich, we need to know that w is $\Box^{k+1}$-P-inductive. This explains why the infinitely many rules CD(k), $k \in \mathbb{N}$, are needed.

Definition 17. *Let us consider the frame $\langle G^{\mathsf{L}}, R\rangle$ where:*

- *G^{L} is the class of all $\mathcal{L}^P$-saturated sets of formulas of $\mathcal{L}^P$ for some denumerable set of parameters P;*
- *wRv iff $\Box^-(w) \subseteq v$.*

This frame is likely to be composed of a number of parts, each completely isolated from any of the others. Such frames are said to be *non-cohesive*. Following [6, p. 78], a *cohesive* frame is one in which, for every w and $w' \in W$, $w(R \cup R^{-1})^n w'$ for some $n \geq 0$, where $w(R \cup R^{-1})w'$ means that either wRw' or $w'Rw$.

Definition 18 (Normal canonical model)**.** *A* normal canonical model *for* L *is a tuple* $\mathcal{M}^{\mathsf{L}} = \langle W^{\mathsf{L}}, R, D_O, D_L, D_I, V\rangle$*, where:*

- *$\langle W^{\mathsf{L}}, R\rangle$ is one of the cohesive frames of which $\langle G^{\mathsf{L}}, R\rangle$ is composed;*
- *$D_O = \{[p]_w :$ for some $w \in W^{\mathsf{L}}$, where $p \in P \cup OBJ$;*
- *$D_L = \{\ell_{\widehat{f}} : f \in INT\}$;*
- *$D_I = \{\widehat{f} : \widehat{f}(w) = \langle [p]_w, \ell_{\widehat{f}}\rangle$ iff $\langle \lambda x.x = p\rangle f \in w$ and $\widehat{f}(w) = \langle \ell_{\widehat{f}}, \ell_{\widehat{f}}\rangle$ iff $\langle \lambda x.x = p\rangle f \notin w$ for all $p \in P \cup OBJ\}$.*
- *the valuation V is a function with domain W^{L} that is such that:*
  $V(P^{n,m}, w) = \{\langle [p_1]_w, ..., [p_n]_w, \widehat{f_1}, \dots, \widehat{f_m}\rangle : Pp_1, ..., p_n, f_1, \dots, f_m \in w\}$.
  $V(=^{2,0}, w) = \{\langle [p]_w, [p]_w\rangle : p \in P \cup OBJ\}$.

Remark 19. *The frame $\langle W^{\mathsf{L}}, R\rangle$ is cohesive and so for every $w, v \in W^{\mathsf{L}}$, $[p]_w = [p]_v$ thanks to axioms (9) and (10). So we can write $[p]$ instead of $[p]_w$.*

As to the interpretation $\widehat{f}$ of f, the equivalence class $[p]$ is unique since by Lemma 3.3, $\vdash_{\mathsf{L}} [\langle \lambda y.p = y\rangle f \wedge \langle \lambda y.p' = y\rangle f] \rightarrow (p = p')$.

We avoid the problem noted in [5] of two distinct intensional variables f and g that have the same graph – i.e., $(\langle \lambda x.x = p\rangle f \in w)$ iff $(\langle lx.x = p\rangle g \in w)$, for all $w \in W^{\mathsf{L}}$ – but satisfy different formulas.[6] *The labels prevent from having $\widehat{f} = \widehat{g}$. In fact if $f \not\equiv g$, then $\ell_f \neq \ell_g$ and, therefore f and g will be interpreted to different intensions: $\widehat{f} \neq \widehat{g}$.*

[6]In Fitting [4] f and g are interpreted to the same intension.

Lemma 20 ($\mathsf{D}f$)**.** $\langle\lambda x.\top\rangle f \in w$ *iff* $\widehat{f}(w) = \langle [p]_w, \ell_{\widehat{f}}\rangle$ *for some* $p \in P \cup OBJ$.

Proof. Since w is $\Diamond^0$-P-rich if $\langle\lambda x.\top\rangle f \in w$ then there is a $p \in P \cup OBJ$ such that $\langle\lambda x.x = p\rangle f \in w$, therefore $\widehat{f}(w) = \langle [p]_w, \ell_{\widehat{f}}\rangle$, see Remark 9. If $\widehat{f}(w) = \langle [p]_w, \ell_{\widehat{f}}\rangle$ for some $p \in P \cup OBJ$, then $\langle\lambda x.x = p\rangle f \in w$ and so $\langle\lambda x.\top\rangle f \in w$ since w is deductively closed and $\vdash_{\mathsf{FOIL}} \langle\lambda x.x = p\rangle f \to \langle\lambda x.\top\rangle f$. □

Lemma 21 (Assignments and substitutions)**.** *Let $\mathcal{M}^{\mathsf{L}}$ be a normal canonical model for* L *and let σ be the assignment such that $\sigma(p) = [p]$ and $\sigma(f) = \widehat{f}$. For all $w \in W^{\mathsf{L}}$ and for all formula A of $\mathcal{L}^P$,*

$$\sigma^{x \rhd [p]} \models_w^{\mathcal{M}^{\mathsf{L}}} A \quad \textit{iff} \quad A[p/x] \in w.$$

Proof. By induction on A. We consider two cases.

$\sigma^{x \rhd [p]} \models_w^{\mathcal{M}^{\mathsf{L}}} \langle\lambda y.y = x\rangle f$ iff by Def. 1 and def. of D_O, for some $s \in P \cup OBJ$,

$\sigma^\star(f)(w) = [s]$	and	$\sigma^{x \rhd [p], y \rhd [s]} \models_w y = x$	iff	by Def.1
$\sigma^\star(f)(w) = [s]$	and	$\langle [s], [p]\rangle \in V(=, w)$	iff	by def. of V(=, w)
$s^\star(f)(w) = [s]$	and	$(s = p) \in w$	iff	by def. of $\sigma^\star$
$\sigma(f)(w) = \langle [s], \ell_{\widehat{f}}\rangle$	and	$(s = p) \in w$	iff	by def. of D_I
$\langle\lambda y.y = s\rangle f \in w$	and	$(s = p) \in w$	iff	by Lemma 3.4
$\langle\lambda y.y = p\rangle f \in w$				

$\sigma^{x \rhd [p]} \models_w^{\mathcal{M}^{\mathsf{L}}} \langle\lambda y.P(y, x)\rangle f$ iff by Def. 1 and def. of D_O for some $s \in P \cup OBJ$,

$\sigma^\star(f)(w) = [s]$	and	$\sigma^{x \rhd [p], y \rhd [s]} \models_w P(y, x)$	iff	by Def. 1
$\sigma^\star(f)(w) = [s]$	and	$\langle [s]\,[p]\rangle \in V(P, w)$	iff	by def. of $V(P, w)$
$\sigma^\star(f)(w) = [s]$	and	$P(s, p) \in w$	iff	by def. of $\sigma^\star$
$\sigma(f)(w) = \langle [s], \ell_{\widehat{f}}\rangle$	and	$P(s, p) \in w$	iff	by def. of D_I
$\langle\lambda y.y = s\rangle f \in w$	and	$P(s, p) \in w$	iff	by Lemma 3.5
$\langle\lambda y.P(y, p)\rangle f \in w$				

□

Lemma 22 (Truth lemma)**.** *Let $\mathcal{M}^{\mathsf{L}}$ be a normal canonical model for* L *and let σ be the assignment such that $\sigma(p) = [p]$ and $\sigma(f) = \widehat{f}$. For all $w \in W^{\mathsf{L}}$ and for all formula A of $\mathcal{L}^P$,*

$$\sigma \models_w^{\mathcal{M}^{\mathsf{L}}} A \quad \textit{iff} \quad A \in w$$

Proof. By induction on A, we consider two cases.

$\sigma \models_w \langle\lambda x.B\rangle f$ iff by Def. 1 and def. of D_O there is a $p \in P \cup OBJ$ such that

$\sigma^\star(f)(w) = [p]$	and	$\sigma^{x \rhd [p]} \models_w B$	iff	by Lemma 21
$\sigma^\star(f)(w) = [p]$	and	$\sigma \models_w B[p/x]$	iff	by induction hypothesis
$\sigma^\star(f)(w) = [p]$	and	$B[p/x] \in w$	iff	by definition of $\sigma^\star$
$\sigma(f)(w) = \langle [p], \ell_{\widehat{f}}\rangle$	and	$B[p/x] \in w$	iff	by definition of D_I
$\langle\lambda x(x = p)f \in w$	and	$B[p/x] \in w$	iff	by Lemma 3.5
$\langle\lambda x.B\rangle f \in w$				

$\sigma \models_w \Box B$	iff by Def. 1
for all $v \in W^{\mathsf{L}}, wRv, \sigma \models_v B$	iff by induction hypothesis
for all $v \in W^{\mathsf{L}}, wRv, B \in v$	iff by Lemma 14
$\Box B \in w$.	

□

Theorem 23. *Let* S *be any of the propositional modal logics of figure 1. Each* FOIL.CD.S*-consistent set of formulas* Δ *has a model with constant domains based on a frame for* S.

Proof. Let L = FOIL.CD.S be any of the logics referred to in the theorem. By Lemma 12, the FOIL.CD.S-consistent set Δ can be extended to an L^P-saturated set of formulas $\Delta^\star$ for some denumerable set P of parameters. $\Delta^\star$ is a member of at least a cohesive frame of which $\langle G^{\mathsf{L}}, R\rangle$ is composed of. Take a normal canonical model $\mathcal{M}^{\mathsf{L}}$ based on such a cohesive frame and constructed according to Definition 18.

By the truth Lemma 22, there is a world w and an assignment σ s.t.: $\sigma \models_w^{\mathcal{M}^{\mathsf{L}}} D$ for each $D \in \Delta$ – i.e., $\mathcal{M}^{\mathsf{L}}$ is a model of Δ.

Moreover, $\mathcal{M}^{\mathsf{L}}$ is a model for the logic L^P because every theorem of L^P is true at every world, i.e. it is satisfied at every world by all the assignments. To wit, let $\vdash_{\mathsf{L}^P} A(\vec{x}, \vec{f})$ and let τ be an assignment such that $\tau(x_i) = p_i$ and $\tau(f_i) = \widehat{f_i}$. Then $\vdash_{\mathsf{L}^P} A(\vec{p}, \vec{\widehat{f}})$ and $A(\vec{p}, \vec{\widehat{f}}) \in w$ because w is deductively closed. By Lemma 22 , $\sigma \models_w^{\mathcal{M}^{\mathsf{L}}} A(\vec{p}, \vec{\widehat{f}})$ and, therefore, $\tau \models_w^{\mathcal{M}^{\mathsf{L}}} A(\vec{x}, \vec{f})$.

The proof that $\mathcal{M}^{\mathsf{L}}$ is based on a frame for S follows the usual pattern, as for the propositional base. □

For logics containing axiom *CI*, it is immediate to acknowledge that $\mathcal{M}^{\mathsf{L}}$ is based on a CI-frame since, by closure under *CI*, $\mathsf{D}f \in w$ for all $w \in W^{\mathsf{L}}$ and all intensional variables f.

7 Conclusion

One problem that remains open is the status of the CD-rule (for $k > 0$) when B is not a theorem. Given Fitting's [4] completeness result, we know that the CD-rule is semantically admissible in FOIL.S when S is one of K, D, T, K4, and S4. Nevertheless, it would be interesting to know if the CD-rule is also derivable or only admissible in these calculi (as well as in calculi including axiom 5).

Another problem that remains open concerns a complete axiomatisation of constant domains FOIL over a quantifier-free and identity-free language. As shown in [2] the S5-based FOIL over this language is not decidable and without identity we don't know how to state anything like the CD-rule.

One line of future research is the extension of FOIL with an identity predicate between intensions $\approx$. In particular, the addition of labels to intensions shall allow a fine-grained treatment of intensional identity where we can distinguish two intensions which coincide as to the objects designated in every world: it could be that $\widehat{f}(w) = \langle o, \ell_{\widehat{f}}\rangle$ iff $\widehat{g}(w) = \langle o, \ell_{\widehat{g}}\rangle$ for all $w \in W$ and $o \in D_O$.

References

[1] Corsi, G. (2002). A unified completeness theorem for quantified modal logics. *The Journal of Symbolic Logic*, 67, 4, pp. 1483−1510.

[2] Fitting, M. (2002). Modal logics between propositional and first-order. *Journal of Logic and Computation*, 12, 6, pp. 1017−1026.

[3] Fitting, M. (2004). First-order intensional logic. *Annals of Pure and Applied Logic*, 127, 1−3, pp. 171−193.

[4] Fitting, M. (2006). FOIL axiomatized. *Studia Logica*, 84, 1, pp. 1−22.

[5] Fitting, M. (2007). Correction to 'FOIL axiomatized'. *Studia Logica*, 85, 1, p. 275.

[6] Hughes, G. E. and Cresswell, M. J. (1984). *A Companion to Modal Logic*. London: Methuen.

[7] Hughes, G. E. and Cresswell, M. J. (1996). *A New Introduction to Modal Logic*. London: Routledge.

[8] Stalnaker, R. and Thomason, R. H. (1968). Abstraction in first-order modal logic. *Theoria*, 34, 1, pp. 203−207.

[9] Thomason, R. H. (1970). Some completeness results for modal predicate calculi. In Karel Lambert, (ed.) *Philosophical Problems in Logic: Some Recent Developments*. Dordrecht: Reidel. pp. 56−76.

[10] Thomason R. H. and Stalnaker, R. (1968). Modality and reference. *Nous*, 2, 1, pp. 359–372.

[11] Wang, Y. and Fan, J. (2014). Conditionally knowing what. In Goré, R., Kooi, B. and A. Kurucz, (eds). *Advances in Modal Logic 10*. London: College Publications. pp. 569–587.

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